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MORE ON $*_*$-CONNECTEDNESS

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Abstract. In [4] $*_*$-connected, $*$-cl-connected and $*$ -cl*-connected ideal space are introduced and studied by Modak and Noiri. We further study the properties of these sets and give a characterization of $*_*$-connected ideal space.

The concept of ideal topological space has been studied by Kuratowski [3] Kuratowski and Vaidyanathswamy, [5]. A nonempty collection I of subsets of X is called an *ideal* in X if it has the following properties:

(i) $A \in I$ and $B \subset A$ implies $B \in I$ and

(ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I in X, (X, τ, I) is an *ideal topological space* (we call it an *ideal space*). For a subset A of X the local function of A is defined as follows, [3]:

$$A^*(I, \tau) = \{x \in X \mid U \cap A \notin I\},$$

for every $U \in \tau(x)$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of basic facts about the local function [2] without mentioning it explicitly. A Kuratowski closure operator $cl^*()$ for a topology $\tau^*(I, \tau)$, called the $*$-*topology finer then* τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$, [5]. When there is no room for confusion, we simply write A^* for $A^*(\tau, I)$ and τ^* for $\tau^*(I, \tau)$. $*_*$-*connected*, $*$-*cl-connected* and $*$-*cl*-connected* ideal spaces are introduced and studied by Modak and Noiri in [4]. In this paper we further study the properties of these spaces.

If $A \subset X$, clA and $int(A)$ will denote the *closure* and *interior* of A in (X, τ) and $int^*(A)$ will denote the *interior* of A in (X, τ^*). Subsets of X closed in (X, τ^*) are called $*$-*closed* sets. A subset A of X in ideal space (X, τ, I) is $*$-closed if and only if $A^* \subset A$, [2]. Nonempty subsets A, B of an ideal space (X, τ, I) are called $*_*$-*separated* if $A^* \cap B = A \cap B^* = A \cap B = \emptyset$ and are called $*$-*cl-separated* (resp. $*$-*cl*-separated*) if $A^* \cap cl(B) = cl(A) \cap B^* = A \cap B = \emptyset$ (resp. $A^* \cap cl^*(B) = cl^*(A) \cap B^* = A \cap B = \emptyset$), [4].

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A subset A of an ideal space (X, τ, I) is called $*_*$-*connected* if A is not the union of two $*_*$-separated sets in (X, τ, I) and is called $*$-*cl-connected* (resp. $*$ -cl^*-*connected*) if A is not the union of two $*$-cl-separated (resp. $*$-cl*-separated sets in (X, τ, I)), [4].

1. Main Results

Lemma 1. [4] *Let (X, τ, I) be an ideal space. If A is a $*_*$-connected set of X and H, G are $*_*$-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Lemma 2. [4] *Let (X, τ, I) be an ideal space. If A is a $*$-cl-connected subset of X and H, G are $*$ -cl-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Lemma 3. [4] *Let (X, τ, I) be an ideal space. If A is a $*$-cl*-connected subset of X and H, G are $*$-cl*-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Theorem 1. *Let (X, τ, I) be an ideal space. If A and B are nonempty disjoint sets such that A and B are $*$ -open, then A and B are $*_*$-separated.*

Proof. Let A and B be nonempty disjoint sets such that $A \cap B = \emptyset$. We have $A \subset X - B$ and so $cl^*(A) \subset cl^*(X - B) = X - B$. Always $A^* \subseteq cl^*(A) \subset X - B$ which implies that $A^* \cap B = \emptyset$. Again $B \subset X - A$ which implies that $A \cap B^* = \emptyset$. So, $A^* \cap B = A \cap B^* = A \cap B = \emptyset$ and therefore A and B are $*_*$-separated. □

Theorem 2. *Let A and B be two $*$-cl-separated sets in an ideal space (X, τ, I). If C and D are nonempty subsets such that $C \subset A$ and $D \subset B$, then C and D are also $*$-cl-separated.*

Proof. Since A and B are $*$-cl-separated, $A^* \cap cl(B) = cl(A) \cap B^* = A \cap B = \emptyset$. We know that, since $C \subset A$ and $D \subset B$, then $cl(C) \subset cl(A)$, $cl(D) \subset cl(B)$ and $C^* \subset A^*$, $D^* \subset B^*$ such that $C^* \cap cl(D) \subset A^* \cap cl(B) = \emptyset$ and so $C^* \cap cl(D) = \emptyset$. Similarly we can prove that $cl(C) \cap D^* = \emptyset$. Hence, $C^* \cap cl(D) = cl(C) \cap D^* = C \cap D = \emptyset$. Consequently, C and D are $*$-cl-separated. □

Theorem 3. *Let A and B be two $*_*$-separated (resp. $*$-cl*-separated) sets in an ideal space (X, τ, I). If C and D are nonempty subsets such that $C \subset A$ and $D \subset B$, then C and D are also $*_*$-separated (resp. $*$-cl*-separated).*

Proof. The proof is similar with the proof of Theorem 2. □

Theorem 4. *Let (X, τ, I) be an ideal space and $A, B \subset X$. If A and B are $*$-open or $*$-closed, then $A - B$ and $B - A$ are $*_*$-separated.*

Proof. $(A-B)^* \cap (B-A) \subseteq (A \cap (X-B))^* \cap (B-A) \subset A^* \cap (X-B)^* \cap (B \cap (X-A)) = A^* \cap (X-A) \cap (X-B)^* \cap B$. If A is $*$-closed, then $A^* \subset A$, such that $A^* \cap (X-A) \cap (X-B)^* \cap B = \emptyset$. If B is $*$-open, $(X-B)$ is $*$-closed, then $(X-B)^* \subset (X-B)$, such that $A^* \cap (X-A) \cap (X-B)^* \cap B = \emptyset$. Similarly, we can show that $(A-B) \cap (B-A)^* = \emptyset$. Consequently, $A-B$ and $B-A$ are $*_*$-separated. □

Theorem 5. *Let a and b be distinct points of a subset C of an ideal space (X, τ, I). If they are elements of some $*_*$-connected subset of C, then C is a $*_*$-connected subset of X.*

Proof. Suppose C is not $*_*$-connected. Then there exist nonempty subsets A and B of X such that $A^* \cap B = A \cap B^* = A \cap B = \emptyset$ and $C = A \cup B$. Since A and B are nonempty sets there exists a point $a \in A$ and a point $b \in B$. By hypothesis, a and b must be elements of a $*_*$-connected subset E of C. Since $E \subset A \cup B$, by Lemma 1, either $E \subset A$ or $E \subset B$. Consequently, either a and b are both in A or both in B. Let's say that a and b are elements of the set A. Then $A \cap B \neq \emptyset$, that is a contradiction to the fact that A and B are disjoint. Therefore, C must be $*_*$-connected. □

Theorem 6. *Let a and b be distinct points of a subset C of an ideal space (X, τ, I). If they are elements of some $*$-cl-connected (resp. $*$-cl*-connected) subset of C, then C is a $*$-cl-connected (resp. $*$-cl*-connected) subset of X.*

Proof. The proof is similar with the proof of Theorem 5. □

Theorem 7. *Let (X, τ, I) be a $*_*$-connected ideal topological space. If A is a $*_*$-connected subset of (X, τ, I) and $X - A$ is a union of two $*_*$-separated sets B and C, then $A \cup B$ and $A \cup C$ are $*_*$-connected.*

Proof. Suppose that $A \cup B$ is not $*_*$-connected. Then there exist two nonempty $*_*$-separated sets G and H, such that $A \cup B = G \cup H$. Since A is a $*_*$-connected, $A \subset A \cup B = G \cup H$. From Lemma 2 we know either $A \subset H$ or $A \subset G$. Suppose $A \subset G$. Since $A \cup B = G \cup H$, $A \subset G$ implies that $A \cup B \subset G \cup B$ and so $G \cup H \subset G \cup B$. Hence, $H \subset B$. Since B and C are $*_*$-separated, H and C are also $*_*$-separated. Thus H is $*_*$-separated from G as well as C. Now, $H^* \cap (G \cup C) = (H^* \cap G) \cup (H^* \cap C) = \emptyset$, $H \cap (G \cup C)^* = H \cap (G^* \cup C^*) = (H \cap G^*) \cup (H \cap C^*) = \emptyset$ and $H \cap (G \cup C) = (H \cap G) \cup (H \cap C) = \emptyset$. Therefore, H is $*_*$-separated from $G \cup C$. Since $X - A = B \cup C$, $X = A \cup (B \cup C) = (A \cup B) \cup C = (G \cup H) \cup C$, $A \cup B = G \cup H$ and so $X = (G \cup C) \cup H$. Thus, X is the union of two nonempty $*_*$-separated sets $G \cup C$ and H, which is a contradiction. Similarly, contradiction will arise if $A \subset H$. Hence $A \cup B$ is $*_*$-connected. One can prove in a similar way that $A \cup C$ is $*_*$-connected. □

Theorem 8. *If A is a $*$-cl-connected ($*$-cl*-connected) subset of a $*$-cl-connected ($*$-cl*-connected) ideal topological space (X, τ, I) such that $X - A$ is a union of two $*$-cl-separated ($*$-cl*-separated) sets B and C then $A \cup B$ and $A \cup C$ are $*$-cl-connected ($*$-cl*-connected).*

Proof. The proof is similar with the proof of Theorem 7. □

Theorem 9. *Let A and B be two $*_*$-connected sets of an ideal space (X, τ, I). If they are not $*_*$-separated, then $A \cup B$ is $*_*$-connected.*

Proof. Let A and B be $*_*$-connected in X. Suppose $A \cup B$ is not $*_*$-connected. Then, there exist two nonempty disjoint $*_*$-separated sets G and H such that $A \cup B = G \cup H$. Since A and B are $*_*$-connected by Lemma 1 either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$. Now, if $A \subset G$ and $B \subset H$, then $A \cap H = B \cap G = \emptyset$. Therefore $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (A \cap G) \cup \emptyset = (A \cap G) = A$. Also, $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = \emptyset \cup (B \cap H) = (B \cap H) = B$. Similarly, if $B \subset G$ and $A \subset H$ then $(A \cup B) \cap G = A$ and $(A \cup B) \cap H = B$. Now, $((A \cup B) \cap H) \cap ((A \cup B) \cap G)^* \subset ((A \cup B) \cap H) \cap ((A \cup B)^* \cap G^*) = (A \cup B) \cap H \cap G^* \cap (A \cup B)^* = \emptyset$, $((A \cup B) \cap H)^* \cap ((A \cup B) \cap G) \subset ((A \cup B)^* \cap H^*) \cap ((A \cup B) \cap G) = (A \cup B)^* \cap H^* \cap G \cap (A \cup B) = \emptyset$ and $((A \cup B) \cap H) \cap ((A \cup B) \cap G) = ((A \cup B) \cap H \cap G \cap (A \cup B) = \emptyset$. Therefore, $(A \cup B) \cap G$ and $(A \cup B) \cap H$ are $*_*$-separated. Thus, A and B are $*_*$-separated which is a contradiction. Hence, $A \cup B$ is $*_*$-connected. □

Theorem 10. *Let A and B be two $*$-cl-connected sets of an ideal space (X, τ, I). If none of them is $*$-cl-connected, then $A \cup B$ is $*$-cl-connected.*

Proof. Let A and B be $*$-cl-connected sets in X. Suppose $A \cup B$ is not $*$-cl-connected. Then, there exist two nonempty disjoint $*$-cl-separated sets G and H such that $A \cup B = G \cup H$. Since A and B are $*$-cl-connected, by Lemma 1 either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$. Now, if $A \subset G$ and $B \subset H$, then $A \cap H = B \cap G = \emptyset$. Therefore $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (A \cap G) \cup \emptyset = (A \cap G) = A$. Also $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = \emptyset \cup (B \cap H) = (B \cap H) = B$. Similarly, if $B \subset G$ and $A \subset H$, then $(A \cup B) \cap G = A$ and $(A \cup B) \cap H = B$. Now, $((A \cup B) \cap H)^* \cap cl((A \cup B) \cap G) \subset ((A \cup B)^* \cap H^*) \cap cl(A \cup B) \cap clG = (A \cup B)^* \cap cl(A \cup B) \cap H^* \cap clG = \emptyset$, $cl((A \cup B) \cap H) \cap ((A \cup B) \cap G)^* \subset cl(A \cup B) \cap clH \cap (A \cup B)^* \cap G^* = cl(A \cup B) \cap (A \cup B)^* \cap clH \cap G^* = \emptyset$ and $((A \cup B) \cap H) \cap ((A \cup B) \cap G) = ((A \cup B) \cap H \cap G \cap (A \cup B) = \emptyset$. Therefore, $(A \cup B) \cap G$ and $(A \cup B) \cap H$ are $*$-cl-separated. Thus, A and B are $*$-cl-separated which is a contradiction. Hence, $A \cup B$ is $*$-cl-connected. □

Theorem 11. *If A and B are $*$-cl*-connected sets of an ideal space (X, τ, I) such that none of them is $*$-cl*-separated, then $A \cup B$ is $*$-cl*-connected.*

Proof. The proof is similar with the proof of Theorem 10. □

The following example shows that the union of two $*_*$-connected, (resp. $*$-cl-connected, $*$-cl^*-connected) sets is not a $*_*$-connected set, (resp. $*$-cl-connected, $*$-cl^*-connected sets). However, Theorems 9, 10 and 11 show that the union of two $*_*$-connected (resp. $*$-cl-connected, $*$-cl^*-connected sets is a $*_*$-connected, (resp. $*-cl$-connected, $*-cl^*$-connected) set if none of them is $*_*$-separated (resp. $*-cl$-separated, $*-cl^*$-separated).

Example 1. Let (X, τ, I) be ideal space, where $X = \{a, b, c, d\}$, $\tau = \{\varnothing, \{b\}, \{b, c\}, \{a, b, d\}, X\}$ and $I = \{\varnothing, \{b\}\}$. If $A = \{a, b\}$, $B = \{a, d\}$, then A and B are $*_*$-connected, but $A \cup B$ is not.

In what follows, $Bd(A)$ denotes the of boundary A.

Theorem 12. *Let (X, τ, I) be an ideal space and $A \subset X$. If C is $*_*$-connected subspace of X that intersect both A and $X - A$, then C intersects $Bd(A)$.*

Proof. Suppose $C \cap Bd(A) = \emptyset$. So, $Bd(A) = (cl(A) \cap cl(X - A))$ and $C \cap (cl(A) \cap cl(X - A)) = \emptyset$. Now, $C = C \cap X = C \cap (A \cup (X - A)) = (C \cap A) \cup (C \cap (X - A))$. Also, we know that $(C \cap A)^* \cap (C \cap (X - A)) \subset (C^* \cap A^*) \cap (C \cap (X - A)) = C \cap C^* \cap (A^* \cap (X - A))$. So, $A^* \subset cl(A)$ and $(X - A) \subset cl(X - A) = \emptyset$, which implies that $C^* \cap C \cap (A^* \cap (X - A)) = \emptyset$. Then $(C \cap A) \cap (C \cap (X - A))^* \subset (C \cap A) \cap (C^* \cap (X - A)^*) = C \cap C^* \cap (A \cap (X - A)^*) = \emptyset$ and $(C \cap A) \cap (C \cap (X - A)) = C \cap A \cap (X - A) = \emptyset$. Thus, $(C \cap A)$ and $(C \cap (X - A))$ form a $*_*$-separation for C, which is a contradiction. Hence, $C \cap Bd(A) \neq \emptyset$. □

Theorem 13. *Let (X, τ, I) be an ideal space and $A \subset X$. If C is $*$-cl-connected (resp. $*$-cl^*-connected) subspace of X that intersect both A and $X - A$, then C intersects $Bd(A)$.*

Proof. The proof is similar with the proof of Theorem 12. □

Theorem 14. *Let (X, τ, I) be an ideal space. Both $*$-closed or $*$-open separated sets in this space are $*_*$-separated.*

Proof. A and B are separated sets, $A \cap B = \emptyset$. If A and B are $*$-closed, then $A^* \subset A$ and $B^* \subset B$, so that $A^* \cap B = A \cap B^* = A \cap B = \emptyset$. We will prove that A and B are $*_*$-separated. If A and B are $*$-open, then $X - A$ and $X - B$ are $*$-closed. So, $A \subset X - B$ which implies that $A^* \subset (X - B)^* \subset X - B$, and $B \subset X - A$ which implies that $B^* \subset (X - A)^* \subset X - A$. We obtain that $A^* \cap B = (X - B) \cap B = \emptyset$ and $A \cap B^* = A \cap (X - A) = \emptyset$. Consequently, A and B are $*_*$-separated. □

Theorem 15. *Let (X, τ, I) be an ideal space and $A, B \subset X$. If $cl^*(A) \cap B = \emptyset$ and $A \cup B$ is $*$-closed, then A is $*$-closed set.*

Proof. So $A \cup B$ is $*$-closed, that implies $cl^*(A \cup B) = cl^*(A) \cup cl^*(B) = A \cup B$ and $cl^*(A) \subset A \cup B$. If $cl^*(A) \cap B = \emptyset$, then $cl^*(A) \subset A$. We know that $A \subset cl^*(A)$ and thus $A = cl^*(A)$. So, A is $*$-closed. □

Theorem 16. *Let (X, τ, I) be an ideal space and $A, B \subset X$. If $A \cap cl^*(B) = \emptyset$ and $A \cup B$ is $*$-open, then A is $*$-open set.*

Proof. If $A \cap cl^*(B) = \emptyset$, then $A \subset (X - cl^*(B))$. Since $A \cup B$ is $*$-open, then $(A \cup B) \cap (X - cl^*(B)) = (A \cap (X - cl^*(B))) \cup (B \cap (X - cl^*(B)))$ is a $*$-open set. So, $A = (A \cap (X - cl^*(B))) \cup (B \cap (X - cl^*(B)))$ is a $*$-open set. □

Corollary 16.1. *Let (X, τ, I) be an ideal space and $A, B \subset X$. If A and B are not $* - cl$-connected (resp. $* - cl^*$-connected) and $A \cup B$ is $*$-closed, then A and B are $*$-closed; if $A \cup B$ is $*$-open, then A and B are $*$-open.*

Tagı, Bhardwaj and Sıngh, in [1] introduced a relationship between continuousness and Cl^*-connectedness and $CL - Cl^*$-connectedness. Now we will examine the relationship between continuousness and $*_*$-connected, (resp. $*$-cl-connected, $*$-cl*-connected) ideal space.

Definition 1. [1] A function $f : (X, \tau_1, I_1) \to (X, \tau_2, I_2)$ is said to be:

(1) *continuous* if the inverse image of each open set in Y under f is open in X;

(2) *contra-continuous* if the inverse image of each open set in Y under f is closed in X;

(3) $\tau_1 - \tau_{2^*}$-*continuous* if the inverse image of each $*$-open set in Y under f is open in X;

(4) $\tau_{1^*} - \tau_{2^*}$-*continuous* if the inverse image of each $*$-open set in Y under f is $*$-open in X;

(5) $\tau_1 - \tau_{2^*}$-*contra-continuous* if the inverse image of each $*$-open set in Y under f is closed in X;

(6) $\tau_{1^*} - \tau_{2^*}$-*contra-continuous* if the inverse image of each $*$-open set in Y under f is $*$-closed in X.

Theorem 17. [1] *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a continuous surjection. If X is $*_*$-connected, then Y is connected.*

Proof. Suppose that Y is not connected. So, there is a nonempty clopen proper subset A of Y. Then $f^{-1}(A)$ is nonempty clopen subset of X and hence, $f^{-1}(A)$ and $X - f^{-1}(A)$ are disjoint. We know that $A^* \subset cl^*(A) \subset cl(A)$. So, $f^{-1}(A)$ is clopen since $(f^{-1}(A))^* \cap (X - f^{-1}(A)) = (f^{-1}(A)) \cap (X - f^{-1}(A))^* = \emptyset$, i.e. X is not $*_*$-connected, that is a contradiction. □

Theorem 18. *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a continuous surjection. If X is resp. $*$-cl-connected, $*$-cl*-connected, then Y is connected.*

Proof. The proof is similar with the proof of Theorem 17. □

Theorem 19. *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a continuous surjection. If X is $*_*$-connected, then Y is $*_*$-connected.*

Proof. Suppose that Y is not $*_*$-connected. Then there are nonempty subsets A and B of Y and hence $Y = A \cup B$, $A^* \cap B = A \cap B^* = A \cap B = \emptyset$. From the fact that f is a continuous surjection, $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A)$, $f^{-1}(B)$ are clopen that $\left(f^{-1}(A)\right)^* \cap f^{-1}(B) = f^{-1}(A) \cap \left(f^{-1}(B)\right)^* = f^{-1}(A) \cap f^{-1}(B) = \emptyset$. We proved that X is not $*_*$-connected, that is a contradiction. □

Theorem 20. *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a continuous surjection. If X is resp. $* - cl$-connected, $* - cl^*$-connected, then Y is resp. $* - cl$-connected, $* - cl^*$-connected.*

Proof. The proof is similar with the proof of Theorem 19. □

Theorem 21. *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a contra-continuous surjection. If X is $*_*$-connected, then Y is connected.*

Proof. Suppose that Y is not connected. Then there are nonempty subsets A and B of Y and hence $Y = A \cup B$, $A^* \cap B = A \cap B^* = A \cap B = \emptyset$. From the fact that f is contra-continuous surjection, $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A)$, $f^{-1}(B)$ are clopen that $\left(f^{-1}(A)\right)^* \cap f^{-1}(B) = f^{-1}(A) \cap \left(f^{-1}(B)\right)^* = f^{-1}(A) \cap f^{-1}(B) = \emptyset$. We proved that X is not $*_*$-connected, that is a contradiction. □

Theorem 22. *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a contra-continuous surjection. If X is $* - cl$-connected, $* - cl^*$-connected, then Y is $* - cl$-connected, $* - cl^*$-connected.*

Proof. The proof is similar with the proof of Theorem 21. □

Theorem 23. *Let $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ be a $\tau_{1^*} - \tau_{2^*}$-continuous surjection. If X is $*_*$-connected, then Y is $*_*$-connected.*

Proof. Suppose that Y is not $*_*$-connected. Then there are nonempty subsets A and B of Y and hence $Y = A \cup B$, $A^* \cap B = A \cap B^* = A \cap B = \emptyset$. Since f is $\tau_{1^*} - \tau_{2^*}$-continuous surjection, then A and B are $*$-open sets in Y so that $f^{-1}(A)$, $f^{-1}(B)$ are $*$-open and $*$-closed in X and so $\left(f^{-1}(A)\right)^* \cap f^{-1}(B) = f^{-1}(A) \cap \left(f^{-1}(B)\right)^* = f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence, X is not $*_*$-connected, that is a contradiction. □

Theorem 24. *Let* $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ *be a* $\tau_{1^*} - \tau_{2^*}$*-continuous surjection. If* X *is* $* - cl^*$*-connected, then* Y *is* $* - cl^*$*-connected.*

Proof. The proof is similar with the proof of Theorem 23 □

Theorem 25. *Let* $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ *be a* $\tau_{1^*} - \tau_{2^*}$*-contra-continuous surjection. If* X *is* $*_*$*-connected and* $* - cl^*$*-connected, then* Y *is* $*_*$*-connected and* $* - cl^*$*-connected.*

Proof. The proof is similar with the proof of Theorem 23 □

Theorem 26. *Let* $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ *be a* $\tau_1 - \tau_{2^*}$*-continuous surjection. If* X *is* $*_*$*-connected, then* Y *is* $*_*$*-connected.*

Proof. We know that $\tau_1 \subset \tau_1^*$ and $\tau_2 \subset \tau_2^*$. If $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ is $\tau_1 - \tau_{2^*}$-continuous, then f is $\tau_{1^*} - \tau_{2^*}$-continuous. And so, via Theorem 25, Y is $*_*$-connected. □

Theorem 27. *Let* $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ *be a* $\tau_1 - \tau_{2^*}$*-continuous surjection. If* X *is* $* - cl^*$*-connected, then* Y *is* $* - cl^*$*-connected.*

Proof. The proof is similar with the proof of Theorem 26. □

Theorem 28. *Let* $f : (X, \tau_1, I_1) \to (Y, \tau_2, I_2)$ *be a* $\tau_1 - \tau_{2^*}$*-contra-continuous surjection. If* X *is* $*_*$*-connected and* $* - cl^*$*-connected, then* Y *is* $*_*$*-connected and* $* - cl^*$*-connected.*

Proof. The proof is similar with the proof of Theorem 26. □

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