Математички Билтен
18 (XLIV)
1994 (45-47)
Скопје, Македонија

ISSN 0351-336X

ON THE BOUNDARY VALUES OF THE HOLOMORPHIC FUNCTIONS IN THE BALL

Nikola Pandeski

Abstract. In this paper is given a new proof of the Alexandrov [1] and E.Low [2] theorem.

Throughout this paper $a\bar{z}$ means the inner product in C^n, $a\bar{z} = a_1\bar{z}_1+\ldots+a_n\bar{z}_n$. The unit ball in C^n is denoted with B, $S=\partial B$ is its boundary and σ ist the rotation invariant measure on S.

Let ζ_k, $k=1,2,\ldots,m$ be any points in S such that every ζ_k, has nonzero components. Let $\{\lambda_k\}$, $k=1,2,\ldots,m$ be complex numbers in the unit disc U. Put

$$\Phi_k(z) = \frac{\lambda_k - z\overline{\zeta_k}}{1-\overline{\lambda_k}z\overline{\zeta_k}} \frac{|\lambda_k|}{\lambda_k}$$

and define

$$f_m(z) = \prod_{k=1}^{m} \Phi_k(z), \quad m\in N$$

Then $f_m(z)$, $m\in N$, are bounded holomorphic functions in the ball and $|f_m(z)| \le 1$ for every m. We will show that if ψ, $0 < \psi < 1$ is a continuous function on S, then there exist a sequence $\{\lambda_k\}$ in U such that $\{f_m(z)\}$ converge to a holomorphic function f in B uniformly on compact subsets of B, and $|f^*(\zeta)| \le \psi(\zeta)$ a.e. σ on S. Then following Alexandrov[1] it follows that there exist function $F\in H^\infty(B)$ such that $|F^*| = \psi$ a.e. σ.

Theorem. Let ψ be a continuous positive function on S, $0 < \psi < 1$. Then there is a function $F\in H^\infty(B)$, $|F| \le 1$ such that $|F^*| = \psi$ a.e. σ on S.

Proof. Cover S with disjoint family of surfaces A_k, $k=1,\ldots,m$ and choose $\zeta_k \in A_k$ with nonzero components. Let $M=\max_{\zeta\in S}[-\log\psi^2(\zeta)]$.

Choose complex numbers $\{\lambda_k\}$, $\lambda_k = \lambda(\zeta_k)$ in the unit disc of the plane such that:

$$(1-|\lambda_k|^2)^n = (\sigma A_k)^{-1}[1-M\sigma A_k-(\log\psi^2(\zeta_k))\sigma A_k]^{(\sigma A_k)^{-2}}+M\sigma A_k = R_1 \quad (1)$$

Let

$$\Phi_k(z) = \frac{\lambda_k - z\overline{\zeta_k}}{1-\overline{\lambda_k}z\zeta_k}\,\frac{|\lambda_k|}{\lambda_k} \quad (2)$$

and define the functions

$$f_m(z) = \prod_{k=1}^{m} \Phi_k(z),\ m\in N \quad (3)$$

We will show that $f_m \to f$, $f\in H^\infty(B)$ uniformly on the compact subsets of the ball and $|f^*(\zeta)| \le \psi(\zeta)$ a.e. σ on S. Put

$$L_k = \frac{1-z\overline{\zeta_k}}{|1-\overline{\zeta_k}z\zeta_k|^2} \quad (4)$$

Then

$$1-|\Phi_k(z)|^2 = (1-|\lambda_k|^2)L_k \quad (5)$$

and

$$\frac{1-|z|}{1+|z|} \le L_k, \quad z\in B \quad (6)$$

We also use the fact that the Poisson kernel for the ball satisfies

$$\left(\frac{1-|z|}{1+|z|}\right)^n \le P(z,\zeta),\ z\in B,\ \zeta\in S \quad (7)$$

Since $(1-x^2)^n \le 1-x^2$, $|x| \le 1$, $(1-x))^\alpha \ge 1-\alpha x$, $\alpha \ge 1$, $x > -1$, we have

$$\begin{aligned} 1-|\Phi_k(z)|^2 &\ge (1-|\Phi_k(z)|^2)^n = (1-|\lambda_k|^2)L_k^m \ge \left(\frac{1-|z|}{1+|z|}\right)^n R_1 \ge \\ &\ge \left(\frac{1-|z|}{1+|z|}\right)^n(\sigma A_k)^2 - \left(\frac{1-|z|}{1+|z|}\right)^2 M\sigma A_k - P(z,\zeta_k)(\log\psi(\zeta_k))\sigma A_k + \\ &+ \left(\frac{1-|z|}{1+|z|}\right)^n M\sigma A_k = -P(z,\zeta_k)(\log\psi(\zeta_k))\sigma A_k + \quad (8) \\ &+ \left(\frac{1-|z|}{1+|z|}\right)^n(\sigma A_k)^2 \end{aligned}$$

and so

$$\begin{aligned} |\Phi_k(z)|^2 &\le 1+P(z,\zeta_k)(\log\psi^2(\zeta_k))\sigma A_k - \left(\frac{1-|z|}{1+|z|}\right)^n(\sigma A_k)^2 \le \\ &\le \exp\{P(z,\zeta_k)(\log\psi^2(\zeta_k))\sigma A_k - \left(\frac{1-|z|}{1+|z|}\right)^n(\sigma A_k)^2\} \end{aligned} \quad (9)$$

This implies

$$|f_m(z)|^2 \le \exp\{\sum_{k=1}^{m} P(z,\zeta_k)(\log\psi^2(\zeta_k))\sigma A_k - \\ - \sum_{k=1}^{m} (\frac{1-|z|}{1+|z|})^n (\sigma A_k)^2\} \quad (10)$$

Because $\sum_{k=1}^{m} (\sigma A_k)^2 \to 0$ if $m \to \infty$ we have

$$|f(z)|^2 \le \exp\{\int_S P(z,\zeta)(\log\psi^2(\zeta))d\sigma\} \quad (11)$$

and so $|f(\zeta)| \le \psi(\zeta)$ a.e. σ on S.

Put $F(z) = z_1 f(z)$, $z \in B$. Then

a) $F(0) = 0$

b) $|F \zeta| < |\zeta_1|\psi(\zeta)$ on S

c) $\int_S [|\zeta_1|\psi(\zeta) - \mathrm{Re}F(\zeta)]^p d\sigma < \gamma^p \int_S \Phi p(\zeta) d\sigma$ for some γ

d) $||\zeta\psi(\zeta)\ \mathrm{Re}F(\zeta)||\ \ ||\zeta\psi||$

Then following the Alexandrov discution we get the Theorem 1 in [1] (see also Russian issue of [3]).

R E F E R E N C E S

[1] B.Alexandrov: The existence of inner functions in the ball, Math. Sb. 1/8(160)(1982), N^o 2(6), 147-163

[2] E.Low: E construction of inner functions on the unit ball in C, Inv. Math. 67 (1982), 223-229

[3] W.Rudin: Function theory in the unit ball of C, Springer-Verlag, 1980

ЗА ГРАНИЧНИТЕ ВРЕДНОСТИ НА ХОЛОМОРФНИТЕ ФУНКЦИИ ВО ТОПКАТА

Никола Пандески

Р е з и м е

Во работата е даден еден нов доказ на теоремата 1 на Александров [1] и на E.Low [2].

Prirodno-matematički fakultet
p.f. 162
91000 Skopje
Macedonia