Математички Билтен
23 (XLIX)
1999 (47–58)
Скопје, Македонија

ISSN 0351–336X

THE HAHN-BANACH THEOREM FOR BOUNDED n-LINEAR FUNCTIONALS

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Abstract

In [5] was introduced the concept of an n-normed space. In this paper are considered n-functionals and it s connection with bounded linear functionals defined on the factor-space on $(n-1)$-dimensional subspace of a n-normed space and analogy of the Hahn-Banach theorem for the bounded linear n-functionals.

1. Introduction

In this work with L we will denote the n-normed space, in which the n-norm is introduced in [5] as follows:

Let L be a real vector space with dimension greater than n, $n > 1$ and $\|\cdot,\dots,\cdot\|$ is a real function on L^n with the following properties:

i) $\|x_1,\dots,x_n\| \geq 0$, for every $x_1,\dots,x_n \in L$ and $\|x_1,\dots,x_n\| = 0$ if and only if the set $\{x_1,\dots,x_n\}$ is linearly dependent;

ii) $\|x_1,\dots,x_n\| = \|\pi(x_1),\dots,\pi(x_n)\|$ for every $x_1,\dots,x_n \in L$ and every bejektion $\pi\colon \{x_1,\dots,x_n\} \to \{x_1,\dots,x_n\}$;

iii) $\|\alpha x_1,\dots,x_n\| = |\alpha| \cdot \|x_1,\dots,x_n\|$, for every $x_1,\dots,x_n \in L$ and every $\alpha \in R$;

iv) $\|x_1 + x_1',\dots,x_n\| \leq \|x_1,\dots,x_n\| + \|x_1',\dots,x_n\|$, for every $x_1,\dots,x_n$, $x_1' \in L$,

The function $\|\cdot,\dots,\cdot\|$ is called an n-norm on L, and $(L^n, \|\cdot,\dots,\cdot\|)$ is called n-normed space.

Some examples of n-normed spaces are given in [1], [2], [4] and [5].

Definition 1. Let X_i, $i = 1, 2, \ldots, n$ and Y are real vector spaces. An n-linear operator $A: X_1 \times \ldots \times X_n \to Y$ is every function $A(x_1, \ldots, x_n)$, $x_i \in X_i$, $i = 1, 2, \ldots, n$, which is linear in every it's argument. If Y is the set of real numbers, then the n-linear operator is called n-linear functional.

It easy to see that the operator (functional) A is a n-linear if and only if

i) $A(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \sum\limits_{\substack{z_i \in \{x_i, y_i\} \\ i = 1, 2, \ldots, n}} A(z_1, z_2, \ldots, z_n)$, and

ii) $A(\alpha_1 x_1, \alpha_2 x_2, \ldots, a_n x_n) = \alpha_1 \alpha_2 \ldots \alpha_n A(x_1, x_2, \ldots, x_n)$, $\alpha_i \in R$, $i = 1, 2, \ldots, n$.

Definition 2. Let L be a n-normed space. We say that the n-functional f with domain $D(f) \subseteq L^n$ is bounded if there is a real constant $k \geq 0$ such that

$$|f(x_1, x_2, \ldots, x_n)| \leq k\|x_1, x_2, \ldots, x_n\|, \text{ for every } (x_1, x_2, \ldots, x_n) \in D(f).$$

If f is a bounded n-functional, we define a norm of f, denoting by $\|f\|$, with

$$\|f\| = \inf\{k \mid |f(x_1, x_2, \ldots, x_n)| \leq k\|x_1, x_2, \ldots, x_n\|, \text{ for every } (x_1, x_2, \ldots, x_n) \in D(f)\}.$$

If f is not bounded n-functional, then by definition we put $\|f\| = +\infty$.

Lemma 1. Let L be a n-normed space and f is a bounded n-functional with domain $D(f) \subseteq L^n$. If

$$x_i = \lambda x_j, \text{ for some } i, j \in \{1, 2, \ldots, n\}, \ i \neq j, \text{ over } (x_1, x_2, \ldots, x_n) \in D(f),$$

than

$$f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) = 0.$$

Proof. [2]. □

2. The Hahn-Banach theorem for the bounded linear n-functionals

Let L be a real vector space and x is a nonzero element of L. We denote with $P(x)$ the subspace generated by the vector x.

Theorem 1. Let $\{x_1, \ldots, x_{n-1}\}$ be a linear independent subset of the n-normed space L, M is a subspace of L and f is a bounded linear n-functional with domain $M \times P(x_1) \times \ldots \times P(x_{n-1})$. Then, there is a bounded linear n-functional F with domain $L \times P(x_1) \times \ldots \times P(x_{n-1})$ such that

i) $F(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) = f(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1})$, for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) M \times P(x_1) \times \ldots \times P(x_{n-1})$, and
ii) $\|F\| = \|f\|$.

Proof. Let $x \in L \setminus M$ and $H = P(M \cup \{x\})$. If $y_1, y_2 \in M$, then

$$\begin{aligned} &f(y_1, x_1, \ldots, x_{n-1}) - f(y_2, x_1, \ldots, x_{n-1}) = \\ &= f(y_1 - y_2, x_1, \ldots, x_{n-1}) \leq \|f\| \cdot \|y_1 - y_2, x_1, \ldots, x_{n-1}\| \\ &= \|f\| \cdot \|(y_1 + x) - (y_2 + x), x_1, \ldots, x_{n-1}\| \\ &\leq \|f\|(\|y_1 + x, x_1, \ldots, x_{n-1}\| + \|y_2 + x, x_1, \ldots, x_{n-1}\|). \end{aligned}$$

It means that

$$\begin{aligned} &- \|f\| \cdot \|y_2 + x, x_1, \ldots, x_{n-1}\| - f(y_2, x_1, \ldots, x_{n-1}) \leq \\ &\leq \|f\| \cdot \|y_1 + x, x_1, \ldots, x_{n-1}\| - f(y_1, x_1, \ldots, x_{n-1}). \end{aligned} \tag{1}$$

Hence,

$$\begin{aligned} S = &\sup_{y_2 \in M} \{-\|f\| \, \|y_2 + x, x_1, \ldots, x_{n-1}\| - f(y_2, x_1, \ldots, x_{n-1})\} \\ &\leq \inf_{y_1 \in M} \{-\|f\| \, \|y_1 + x, x_1, \ldots, x_{n-1}\| - f(y_1, x_1, \ldots, x_{n-1})\} = S_1 . \end{aligned}$$

Let r be an arbitrary real number such that $S \leq r \leq S_1$. If we put $y_1 = y_2 = y$ in (1), we get

$$|f(y, x_1, \ldots, x_{n-1}) + r| \leq \|f\| \cdot \|y + x, x_1, \ldots, x_{n-1}\| . \tag{2}$$

We define n-functional $\overline{f}$ on $H \times P(x_1) \times \ldots \times P(x_{n-1})$ with

$$\overline{f}(y + \alpha_1 x, \alpha_2 x_1, \ldots, \alpha_n x_{n-1}) = (\alpha_2 \cdot \ldots \cdot \alpha_n)\big(\alpha_1 r + f(y, x_1, \ldots, x_{n-1})\big)$$

We will prove that $\overline{f}$ is linear and bounded. We have

$$\begin{aligned} &\overline{f}(z_1+w_1, z_2+w_2, \ldots, z_n+w_n) = \overline{f}(y_1+\alpha_1 x+y_2+\beta_1 x, \alpha_2 x_1+\beta_2 x_1, \ldots, \alpha_n x_{n-1}+\beta_n x_{n-1}) \\ &= \overline{f}\big(y_1 + y_2 + (\alpha_1 + \beta_1)x, (\alpha_2 + \beta_2)x_1, \ldots, (\alpha_n + \beta_n)x_{n-1}\big) \\ &= (\alpha_2 + \beta_2) \cdot \ldots \cdot (\alpha_n + \beta_n)\big((\alpha_1 + \beta_1)r + f(y_1 + y_2, x_1, \ldots, x_{n-1})\big) \\ &= \big(\alpha_1 r + f(y_1, x_1, \ldots, x_{n-1})\big) \sum_{\substack{t_i \in \{\alpha_i, \beta_i\} \\ i = 2, \ldots, n}} t_2 \cdot \ldots \cdot t_n + \big(\beta_1 r + f(y_2, x_1, \ldots, x_{n-1})\big) \sum_{\substack{t_i \in \{\alpha_i, \beta_i\} \\ i = 2, \ldots, n}} t_2 \cdot \ldots \cdot t_n \\ &= \sum_{\substack{t_i \in \{\alpha_i, \beta_i\} \\ i = 2, \ldots, n}} \overline{f}(y_1+\alpha_1 x, t_2 x_1, \ldots, t_n x_{n-1}) + \sum_{\substack{t_i \in \{\alpha_i, \beta_i\} \\ i = 2, \ldots, n}} \overline{f}(y_2+\beta_1 x, t_2 x_1, \ldots, t_n x_{n-1}) \\ &= \sum_{\substack{u_i \in \{z_i, w_i\} \\ i=2, \ldots, n}} \overline{f}(z_1, u_2, \ldots, u_n) + \sum_{\substack{u_i \in \{z_i, w_i\} \\ i=2, \ldots, n}} \overline{f}(w_1, u_2, \ldots, u_n) = \sum_{\substack{u_i \in \{z_i, w_i\} \\ i=1, 2, \ldots, n}} \overline{f}(u_1, u_2, \ldots, u_n) \end{aligned}$$

and

$$\overline{f}(\beta_1 z_1, \beta_2 z_2, \dots, \beta_n z_n) = \overline{f}(\beta_1(y+\alpha_1 x), \beta_2\alpha_2 x_1, \dots, \beta_n\alpha_n x_{n-1})$$
$$= \overline{f}(\beta_1 y + \alpha_1\beta_1 x, \alpha_2\beta_2 x_1, \dots,, \alpha_n\beta_n x_{n-1})$$
$$=(\beta_1 \cdot \ldots \cdot \beta_n)(\alpha_1 \cdot \ldots \cdot \alpha_n) r + (\beta_2 \cdot \ldots \cdot \beta_n)(\alpha_2 \cdot \ldots \cdot \alpha_n) f(\beta_1 y, x_1, \dots, x_{n-1})$$
$$= \beta_1 \cdot \ldots \cdot \beta_n(\alpha_1 \cdot \ldots \cdot \alpha_n r + \alpha_2 \cdot \ldots \cdot \alpha_n f(, x_1, \dots, x_{n-1}))$$
$$= \beta_1 \cdot \ldots \cdot \beta_n \overline{f}(y+\alpha_1 x, \alpha_2 x_1, \dots, a_n x_{n-1})$$
$$= \beta_1 \cdot \ldots \cdot \beta_n \overline{f}(z_1, z_2, \dots, z_n),$$

which means that $\overline{f}$ is a linear n-functional with domain $H\times P(x_1)\times\ldots\times P(x_{n-1})$. It is clear that $\overline{f} \equiv f$ on $M \times P(x_1) \times \ldots \times P(x_{n-1})$.

If in (2) we replace y with $\frac{1}{\alpha}y$ where $\alpha \neq 0$ then we get

$$|f(y, x_1, \dots, x_{n-1}) + \alpha r| \leq ||f|| \cdot ||y + \alpha x, x_1, \dots, x_{n-1}||, \text{ for every } \alpha \neq 0$$

which implies

$$|\overline{f}(y + \alpha_1 x, \alpha_2 x_1, \dots, \alpha_n x_{n-1})|$$
$$= |\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_n r + \alpha_2 \cdot \ldots \cdot \alpha_n f(y, x_1, \dots, x_{n-1})|$$
$$= |\alpha_2 \cdot \ldots \cdot \alpha_n| \cdot |\alpha_1 r + f(y, x_1, \dots, x_{n-1})|$$
$$\leq |\alpha_2 \cdot \ldots \cdot \alpha_n| \cdot ||f|| \cdot ||y + \alpha_1 x, x_1, \dots, x_{n-1})||$$
$$= ||f|| \cdot ||y + \alpha_1 x, \alpha_2 x_1, \dots, \alpha_n x_{n-1})||.$$

This means that $\overline{f}$ is a bounded linear n-functional such that $||\overline{f}|| \leq ||f||$. But, $||\overline{f}|| = ||f||$ on $M \times P(x_1) \times \ldots \times P(x_{n-1})$, and so $||\overline{f}|| = ||f||$.

We will consider all pairs $\{X, g\}$, where X is a subspace of L and g is a bounded linear n-functional with domain $X \times P(x_1) \times \ldots \times P(x_{n-1})$. Put $\{X, g\} \prec \{X_1, g_1\}$ if and only if $X \subset X_1$ and g_1 is a extension of g, such that $||g_1|| = ||g||$.

Let T be a subset of all $\{H, \overline{f}\}$ such that $\{M, f\} \prec \{H, \overline{f}\}$. T is a partially ordered set, in which every linear ordered subset has a maximal element. From the Corn Lemma if follows that T has a maximal element $\{K, F\}$. It is clear that $K = L$, since in contrary can be extended in the described way. □

Similar as we prove the theorem 1, we can prove the following corollary:

Corollary 1. Let $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ be a linear independent subset of the n-normed space L, M is a subspace of L and f is a bounded linear n-functional with domain

$$P(x_1) \times \ldots \times P(x_{i-1}) \times M \times P(x_{i+1}) \times \ldots \times P(x_n).$$

Then, there is a bounded linear n-functional

$$F: P(x_1) \times \ldots \times P(x_{i-1}) \times L \times P(x_{i+1}) \times \ldots \times P(x_n) \to R$$

such that

i)
$$F(\lambda_1 x_1, \ldots, \lambda_{i-1} x_{i-1}, x, \lambda_{i+1} x_{i+1}, \ldots, \lambda_n x_n) = \\ = f(\lambda_1 x_1, \ldots, \lambda_{i-1} x_{i-1}, x, \lambda_{i+1} x_{i+1}, \ldots, \lambda_n x_n)$$

for every
$$(\lambda_1 x_1, \ldots, \lambda_{i-1} x_{i-1}, x, \lambda_{i+1} x_{i+1}, \ldots, \lambda_n x_n) \in \\ \in P(x_1) \times \ldots \times P(x_{i-1}) \times M \times P(x_{i+1}) \times \ldots \times P(x_n),$$

ii) $||F|| = ||f||$. □

Remark. *S. Gahler* proved that for $n = 2$ it is not true the general case of Hahn-Banach theorem for bounded linear n-functionals. In other words, for a given bounded linear 2-functional $f: G \times G \to R$, G a subspace of L, in a general case there is no a bounded linear 2-functional $f: L \times L \to R$ such that

$||F|| = ||f||$ and $F(x_1, x_2) = f(x_1, x_2)$, for every $x_1, x_2 \in G$. □

Corollary 2. Let L be an n-normed vector space and $x_1, \ldots, x_n$ is a linear independent subset of L. Then there exist bounded linear n-functionals

$$f_i: P(x_1) \times \ldots \times P(x_{i-1}) \times L \times P(x_{i+1}) \times \ldots \times P(x_n) \to R, \quad i = 1, 2, \ldots, n$$

such that

$$||f_i|| = 1 \quad \text{and} \quad f_i(x_1, \ldots, x_n) = ||x_1, \ldots, x_n||, \quad i = 1, 2, \ldots, n.$$

Proof. It is easy to prove that the mapping

$$f: P(x_1) \times \ldots \times P(x_n) \to R$$

defined by

$$f_i(\lambda_1 x_1, \ldots, \lambda_n x_n) = \lambda_1 \cdot \ldots \lambda_n ||x_1, \ldots, x_n||$$

is a bounded linear n-functional with norm $||f|| = 1$. The Corollary 1 implies that there exist bounded linear n-functionals

$$f_i: P(x_1) \times \ldots \times P(x_{i-1}) \times L \times P(x_{i+1}) \times \ldots \times P(x_n) \to R, \quad i = 1, 2, \ldots, n$$

such that

$$||f_i|| = ||f|| = 1 \text{ and } f_i(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) = ||x_1, \ldots, x_n||$$

for $i = 1, 2, \ldots, n$. □

Definition 3. Let X_i, $i = 1, 2, \ldots, n$ be a real vector spaces. We call the function $p: X_1 \times X_2 \times \ldots \times X_n \to R$.

i) apsolutly homogeneous, if
$p(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) = |\lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n| \, p(x_1, x_2, \ldots, x_n)$,
for every $x_i \in X_i$, $i = 1, 2, \ldots, n$ and every $\lambda_i \in R$, $i = 1, \ldots, n$;

ii) subaditive, if

$$p(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \leq \sum_{\substack{z_i \in \{x_i, y_i\} \\ i = 1, 2, \ldots, n}} p(x_1, x_2, \ldots, x_n),$$

for every $x_i, y_i \in X_i$, $i = 1, 2, \ldots, n$.

Theorem 2. Let L be a real vector space, $p: L^n \to R$ subaditive absolutly homogeneous n-functional, M a subspace of L, $x_2, \ldots, x_n \in L$ and $f: M \times P(x_2) \times \ldots \times P(x_n) \to R$ is a linear n-functional such that

$$f(y, \lambda_2 x_2, \ldots, \lambda_n x_n) \leq p(y, \lambda_2 x_2, \ldots, \lambda_n x_n), \quad \text{for every } y \in M \quad \text{and every} \quad \lambda_i \in R, \; i = 2, \ldots, n.$$

Then, there exist linear n-functional $F: L \times P(x_2) \times \ldots \times P(x_n) \to R$ such that

$$F(x, \lambda_2 x_2, \ldots, \lambda_n x_n) \leq p(x, \lambda_2 x_2, \ldots, \lambda_n x_n), \quad \text{for every } x \in L \quad \text{and every} \quad \lambda_i \in R, \; i = 2, \ldots, n.$$

and

$$F(y, \lambda_2 x_2, \ldots, \lambda_n x_n) = f(y, \lambda_2 x_2, \ldots, \lambda_n x_n), \quad \text{for every } y \in M \quad \text{and every} \quad \lambda_i \in R, \; i = 2, \ldots, n.$$

Proof. Let $x_1 \in L \setminus M$ and $H \doteq P(M \cup \{x_1\})$. For every $y_1, y_2 \in M$ we have

$$\begin{aligned} f(y_1, x_2, \ldots, x_n) - f(y_2, x_2, \ldots, x_n) &= f(y_1 - y_2, x_2, \ldots, x_n) \\ &\leq p(y_1 - y_2, x_2, \ldots, x_n) \\ &= p(y_1 + x_1 - (y_2 + x_1), x_2, \ldots, x_n) \\ &\leq p(y_1 + x_1, x_2, \ldots, x_n) + p(-y_2 - x_1, x_2, \ldots, x_n) \end{aligned}$$

and so

$$\begin{aligned} &- p(-y_2 - x_1, x_2, \ldots, x_n) - f(y_2, x_2, \ldots, x_n) \leq \\ &\leq p(y_1 + x_1, x_2, \ldots, x_n) - f(y_1, x_2, \ldots, x_n). \end{aligned} \tag{1}$$

Hence

$$\begin{aligned} S &= \sup_{y_2 \in M} \{-p(-y_2 - x_1, x_2, \dots, x_n) - f(y_2, x_2, \dots, x_n)\} \\ &\le \inf_{y_1 \in M} \{p(y_1 + x_1, x_2 \dots, x_n) - f(y_1, x_2, \dots, x_n)\} = S_1 . \end{aligned} \tag{2}$$

Let r be an arbitrary real number such that $S \le r \le S_1$. We define n-functional $\overline{f}\colon \times P(x_2) \times \dots \times P(x_n) \to R$ with

$$\overline{f}(y + \lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n) = (\lambda_2, \dots, \lambda_n)(\lambda_1 r + f(y, x_2, \dots, x_n)) .$$

Analogly, we prove the Theorem 1, we can prove that $\overline{f}$ is a linear n-functional. It is clear that $\overline{f} \equiv f$ on $M \times P(x_2) \times \dots \times P(x_n)$.

We will prove that

$$\begin{aligned} &\overline{f}(y + \lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n) \le p(y + \lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n), \\ &\quad \forall y \in M \quad \text{and} \quad \forall \lambda_1, \dots, \lambda_n \in R . \end{aligned} \tag{3}$$

If $\prod_{i=1}^{n} \lambda_i = 0$, then (3) follows from the definition of $\overline{f}$ and the conditions of the theorem. If $\prod_{i=1}^{n} \lambda_i > 0$, then for every $y \in M$ from (1) and (2) follows

$$\begin{aligned} r &\le S_1 \le p\Big(\frac{y}{\lambda_1} + x_1, x_2, \dots, x_n\Big) - f\Big(\frac{y}{\lambda_1}, x_2, \dots, x_n\Big) = \\ &= \frac{1}{\prod_{i=1}^{n} \lambda_i} \big[p(y + \lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n) - f(y, \lambda_2 x_2, \dots, \lambda_n x_n)\big] , \end{aligned}$$

and so

$$r\lambda_1 \cdot \ldots \cdot \lambda_n + \lambda_2 \cdot \ldots \cdot \lambda_n f(y, x_2, \dots, x_n) \le p(y + \lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n),$$

which means that the inequality (3) is true in this case. If $\prod_{i=1}^{n} \lambda_i < 0$, then for every $i \in M$ from (1) and (2) follows

$$\begin{aligned} r &\ge S \ge -p\Big(-\frac{y}{\lambda_1} - x_1, x_2, \dots, x_n\Big) - f\Big(\frac{y}{\lambda_1}, x_2, \dots, x_n\Big) \\ &= -p\Big(\frac{y + \lambda_1 x_1}{-\lambda_1}, x_2, \dots, x_n\Big) - f\Big(\frac{y}{\lambda_1}, x_2, \dots, x_n\Big) \\ &= -\frac{1}{-\prod_{i=1}^{n} \lambda_i} \big[p(y + \lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n) - f(y, \lambda_2 x_2, \dots, \lambda_n x_n)\big] , \end{aligned}$$

and so

$$r\lambda_1 \cdot \ldots \cdot \lambda_n + \lambda_2 \cdot \ldots \cdot \lambda_n f(y, x_2, \ldots, x_n) \leq p(y + \lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n).$$

which means that the inequality (3) is true also in this case.

Now the statement of the theorem, as the proof of Theorem 1, follows from the Corn Lemma. □

3. Connection between bounded linear n-functionals and the linear functionals on the quotient space $L \setminus P(x_1, \ldots, x_{n-1})$

Let $\{x_1, \ldots, x_{n-1}\}$ be a linear independent subset in the n-normed space L. We denote with $P(x_1, \ldots, x_{n-1})$ the subspace of L generated by $\{x_1, \ldots, x_{n-1}\}$, and by L_P the quotient space $L \setminus P(x_1, \ldots, x_{n-1})$. For every $a \in L$ we denote by a_P the class of equivalence of a related to $P(x_1, \ldots, x_{n-1})$. L_P is a vector space with opperations $\alpha a_P = (\alpha a)_P$ and $a_P + b_P = (a + b)_P$. In Lemma 7, [1] was proved that the function $\|\bullet\|_P : L_P \to R$ defined by

$$\|a_P\|_P = \|a, x_1, \ldots, x_{n-1}\|$$

is a norm on the quotient space L_P.

Theorem 3. Let f be a bounded linear n-functional with domain L^n and $\{x_1, \ldots, x_{n-1}\}$ is an arbitrary lineary independent subset of L. The functional $f_P : L_P \to R$ defined with

$$f_P(y_P) = f(y, x_1, \ldots, x_n) \tag{1}$$

is linear, bounded and $\|f_P\| \geq \|f\|$.

Proof. Let a_P, $b_P \in L_P$ and $\lambda \in R$. We have:

$$\begin{aligned} f_P(a_P + b_P) &= f_P\big((a+b)_P\big) = f(a + b, x_1, \ldots, x_{n-1}) \\ &= f(a, x_1, \ldots, x_{n-1}) + f(b, x_1, \ldots, x_{n-1}) = f_P(a_P) + f_P(b_P) \end{aligned}$$

and

$$f_P(\lambda a_P) = f_P\big((\lambda a)_P\big) = f(\lambda a, x_1, \ldots, x_{n-1}) = \lambda f(a, x_1, \ldots, x_{n-1}) = \lambda f_P(a_P),$$

which means that f_P is a linear functional.

Since f is a bounded linear n-functional, there is a real constant $k \leq 0$ such that

$$|f(x_1, x_2, \ldots, x_{n-1})| \leq k\|x_1, x_2, \ldots, x_n\|, \text{ for every } (x_1, x_2, \ldots, x_n) \in L^n.$$

Hence, for every $a_P \in L_P$ it is true that

$$|f_P(a_P)| = |f(a, x_1, \ldots, x_{n-1})| \leq k||a, x_1, \ldots, x_{n-1}|| = k||a_P||_P\,,$$

and so f_P is a bounded functional. It is clear that

$$\begin{aligned}||f_P|| &= \inf\{k|\ |f_P(a_P)| \leq k||a_P||_P, \quad a_P \in L_P\} = \\ &= \inf\{k|\ |f(a, x_1, \ldots, x_{n-1}) \leq k||a, x_1, \ldots, x_{n-1}||\} \geq ||f||\,. \quad \square\end{aligned}$$

The Theorem 3 gives to us the following corollary:

Corollary 3. Let f be a bounded linear n-functional with domain $M \times P(x_1) \times \ldots \times P(x_{n-1})$ were $\{x_1, \ldots, x_{n-1}\}$ is a linear independent set of L, M is a subspace of L and $M_P = \{x_P | x_P \in L_P,\ x \in M\}$. The functional $f_P\colon M_P \to R$ defined by $f_P(y_P) = f(y, x_1, \ldots, x_{n-1})$ is a linear, bounded and $||f_P|| = ||f||$. $\square$

Theorem 4. Let $\{x_1, \ldots, x_{n-1}\}$ be a lineary independent subset of L and $f_P\colon L_P \to R$ be a linear bounded functional. Then, the n-functional

$$f\colon L \times P(x_1) \times \ldots \times P(x_{n-1}) \to R$$

defined by

$$f(a, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = \lambda_1 \ldots \lambda_{n-1} f_P(a_P)$$

is linear, bounded and $||f|| = ||f_P||$.

Proof. Let $(y_1, y_2, \ldots, y_n);\ (z_1, z_2, \ldots, z_n) \in L \times P(x_1) \times \ldots \times P(x_{n-1})$. We have

$$\begin{aligned}
f(y_1+z_1, y_2+z_2, \ldots, y_n+z_n) &= f(y_1+z_1, \lambda_1 x + \mu_1 x_1, \ldots, \lambda_{n-1}x_{n-1} + \mu_{n-1}x_{n-1}) \\
&= f\big(y_1 + z_1, (\lambda_1 + \mu_1)x_1, \ldots, (\lambda_{n-1} + \mu_{n-1})x_{n-1}\big) \\
&= (\lambda_1 + \mu_1) \cdot \ldots \cdot (\lambda_{n-1} + \mu_{n-1}) f_P\big((y_1 + z_1)_P\big) \\
&= (\lambda_1 + \mu_1) \cdot \ldots \cdot (\lambda_{n-1} + \mu_{n-1})\big(f_P(y_{1P}) + f_P(z_{1P})\big) \\
&= \sum_{\substack{t_i \in \{\lambda_i, \mu_i\} \\ i = 1, \ldots, n-1}} t_1 \ldots t_{n-1} f_P(y_{1P}) + \sum_{\substack{t_i \in \{\lambda_i, \mu_i\} \\ i = 1, \ldots, n-1}} t_1 \ldots t_{n-1} f_P(z_{1P}) \\
&= \sum_{\substack{t_i \in \{\lambda_i, \mu_i\} \\ i = 1, \ldots, n-1}} f(y_1, t_1 x_1, \ldots, t_{n-1}x_{n-1}) + \sum_{\substack{t_i \in \{\lambda_i, \mu_i\} \\ i = 1, \ldots, n-1}} f(z_1, t_1 x_1, \ldots, t_{n-1}x_{n-1}) \\
&= \sum_{\substack{u_i \in \{y_i, z_i\} \\ i = 2, \ldots, n}} f(y_1, u_2 \cdot \ldots \cdot u_n) + \sum_{\substack{u_i \in \{y_i, z_i\} \\ i = 2, \ldots, n}} f(z_1, u_2 \cdot \ldots \cdot u_n) = \sum_{\substack{u_i \in \{y_i, z_i\} \\ i = 1, 2, \ldots, n}} f(u_1, u_2 \cdot \ldots \cdot u_n)
\end{aligned}$$

and

$$\begin{aligned} f(\alpha_1 z_1, \alpha_2 z_2, \ldots, \alpha_n z_n) &= f(\alpha_1 z_1, \alpha_1 \lambda_1 x_1, \ldots, \alpha_n \lambda_{n-1} x_{n-1}) \\ &= (\alpha_2 \lambda_1) \cdot \ldots \cdot (\alpha_n \lambda_{n-1}) f_P\big((\alpha_1 z_1)_P\big) \\ &= (\alpha_2 \cdot \ldots \cdot \alpha_n)(\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) f_P(\alpha_1 z_{1_P}) \\ &= (\alpha_1 \alpha_2 \cdot \ldots \cdot \alpha_n)(\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) f_P(z_{1_P}) \\ &= (\alpha_1 \alpha_2 \cdot \ldots \cdot \alpha_n) f(z_1, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \\ &= (\alpha_1 \alpha_2 \cdot \ldots \cdot \alpha_n) f(z_1, z_2, \ldots, z_n) \end{aligned}$$

and so f is a linear n-functional.

Since f_P is a bounded linear functional, there is a real constant $k \geq 0$ such that

$$|f_P(a_P)| \leq k||a_P||_P, \quad \text{for every} \quad a_P \in L_P.$$

So, for every $(z_1, z_2, \ldots, z_n) \in L \times P(x_1) \times \ldots \times P(x_{n-1})$ it is true that

$$\begin{aligned} |f(z_1, z_2, \ldots, z_n)| &= |f(z_1, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})| \\ &= |\lambda_1 \cdot \ldots \cdot \lambda_{n-1} f_P(z_{1_P})| \leq k|\lambda_1 \cdot \ldots \cdot \lambda_{n-1}| \; ||z_{1_P}||_P \\ &= k|\lambda_1, \ldots, \lambda_{n-1}| \; ||z_1, x_1, \ldots, x_{n-1}|| \\ &= k||z_1, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}|| = k||z_1, z_2, \ldots, z_n||, \end{aligned}$$

which means that f is a bounded linear functional. It is clear that:

$$\begin{aligned} ||f|| &= \inf\{k \mid |f(z, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})| \leq k||z, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}||, \ z \in L\} \\ &= \inf\{k \mid |f(z, x_1, \ldots, x_{n-1})| \leq k||z, x_1, \ldots, x_{n-1}||, \ z \in L\} \\ &= \inf\{k \mid |f_P(z_P)| \leq k||z_P||_P, z_P \in L_P\} \\ &= ||f_P||. \quad \square \end{aligned}$$

In the end of this work, using Theorem 4 and Corollary 3, we will present one more proof of the theorem 1.

It is clear that M is an n-normed space, $M_P = \{x_P \mid x_P \in L_P, \ x \in M\}$ is a subspace of L_P. By the corollary 3 $f_P \colon M_P \to R$, defined with

$$f_P(x_P) = f(x, x_1, \ldots, x_{n-1})$$

is a bounded linear functional, such that $||f_P|| = ||f||$. In agree with the Hahn-Banach theorem f_P can be extended to a bounded linear functional

F_P on L_P such that $||F_P|| = ||f_P||$ and $F_P(x_P) = f_P(x_P)$, $\forall x_P \in M_P$. By Theorem 4, the functional $F: L \times P(x_1) \times \ldots \times P(x_{n-1}) \to R$ defined with

$$F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = (\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) F_P(x_P)$$

is a bounded linear functional such that $||F|| = ||F_P|| = ||f_P|| = ||f||$ and for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in M \times P(x_1) \times \ldots \times P(x_{n-1})$ we have

$$\begin{aligned} F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) &= (\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) F_P(x_P) \\ &= (\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) f_P(x_P) \\ &= (\lambda_1 \cdot \ldots \cdot \lambda_{n-1}) f(x, x_1, \ldots, x_{n-1}) \\ &= f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}). \end{aligned}$$

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ТЕОРЕМА НА ХАН-БАНАХ ЗА ОГРАНИЧЕНИ n-ЛИНЕАРНИ ФУНКЦИОНАЛИ

Ристо Малчески

Резиме

Во [5] е воведен поимот за n-нормиран простор. Во оваа работа се разгледани ограничените линеарни n-функционали, нивната врска со ограничените линеарни функционали дефинирани на фактор-простор над $(n-1)$–димензионален подпростор од n-нормиран простор и аналогијата на теоремата на Хан-Банах за ограничени линеарни n-функционали.

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