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UNITARY EQUIVALENCE AND SIMILARITY OF ORDINARY SHIFTS ON OPERATOR VALUED WEIGHTED SEQUENCE SPACES

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Abstract

In this note we consider a number of properties of the operator valued weighted sequence spaces. We discus similarity and unitary equivalence of unilateral unweighted shifts on operator valued weighted sequence spaces.

There is another way of viewing operator valued weighted shifts. We turn to represent operator valued weighted shift on the ordinary space $l^2(H)$ as the ordinary (unilateral unweighted) shift U_+ on a operator valued weighted sequence space $H^2(B)$. The notations in the scalar case suggest these notations. Using this we shell discuss similarity and unitary equivalence of ordinary shifts on operator valued weighted sequence spaces. Let H be a complex separable Hilbert space, $B(H)$ be the algebra of all (linear and bound) operators on H. Let $l^2(H) = \sum\limits_{n=1}^{\infty} \oplus H_n$, $H_n = H$, $\forall n$ be the Hilbert space with an inner product defined by $(f,g) = \sum\limits_{n=1}^{\infty} (f_n, g_n)$ where $f = \sum \oplus f_n$ (or $f = (f_1, f_2, f_3, \ldots)$).

Corresponding to each uniformly bounded sequence $\{A_n\}_{n \in N}$ of bounded and linear operators on H, there is the operator valued weighted unilateral shift A defined on $l^2(H)$ by $A(f_1, f_2, f_3, \ldots) = (0, A_1, f_1, A_2, f_2, \ldots)$ and we shall denote it by $A \sim (A_n)$. With U_+ we shall denote the operator valued weighted shift $U_+ \sim (1)$.

Let $\{B_i\}$ be a sequence of positive, invertible, commuting operators on H with the property $0 < m < B_n < M$, $\forall n \in N$. The space of vectors

$$H^2(B) = \left\{ f = (f_0, \dots, f_n, \dots) \colon f_i \in H,\ i = 0, 1, \dots ; \sum_{i=0}^{\infty} \|B_i f_i\|^2 < \infty \right\}$$

is a Hilbert space with an inner product $(f, g)_B = \sum_{i=0}^{\infty} (B_i f_i, B_i f_i)$ ([1], Proposition 1).

The following theorem which we will need later, is stated here for the reader's convenience. The proof can be found in [1].

Proposition 1. ([1]) *The unilateral shift U_+ on $H^2(B)$ is unitarily equivalent to an operator valued weighted shift with invertible weights $(A_n)_{n=0}^{\infty}$ on $l^2(H)$. Conversely, each operator valued weighted shift with invertible weights on the space $l^2(H)$ is unitarily equivalent to unilateral shift U_+ on a corresponding space $H^2(B)$ for a suitable choice of $(B_n)_{n=0}^{\infty}$. Moreover, the relation between (A_n) and (B_n) is as follows*

$$A_n = B_{n+1} B_n^{-1}, \qquad B_0 = 1, \quad B_n = A_0 A_1 \cdots A_{n-1}.$$

Proposition 2. *Let $\{B_n\}$ and $\{C_n\}$ are sequences of positive, invertible, commuting operators on H with the property $0 < m < B_n < M$ and $0 < l < C_n < L$, $\forall n$, then*

(i) *$H^2(B)$ isometrically isomorphic to $H^2(C)$.*

(ii) *$H^2(B) = H^2(C)$ and the norms are equivalent.*

Proof. (i) Let $V \colon H^2(B) \to H^2(C)$ is defined by

$$V(f_0, f_1 \cdots) = (C_0^{-1} B_0 f_0, C_1^{-1} B_1 f_1, \cdots).$$

It is easy to verify that V is linear and surjective. Also, V is an isometry: $\|Vf\|_C^2 = \sum_{i=1}^{\infty} \|B_i f_i\|^2 = \|f\|_C^2$.

(ii) Let $m, l, M, L, \{B_n\}$ and $\{C_n\}$ be as above and let $f \in H^2(B)$ i.e. $\|f\|_B < \infty$. Then we have

$$\|f\|_C^2 = \sum_{i=0}^{\infty} \|C_i f_i\|^2 = \textstyle\sum_{i=0}^{\infty} \|C_i B_i^{-1} B_i f_i\|^2 \le \frac{L^2}{m^2} \|f\|_B^2 < \infty$$

which means that $f \in H^2(C)$.

Conversely, let $f \in H^2(C)$. Then we have

$$\|f\|_B^2 = \sum_{i=0}^{\infty} \|B_i f_i\|^2 = \sum_{i=0}^{\infty} \|B_i C_i^{-1} C_i f_i\|^2 \le \frac{M^2}{l^2} \sum_{i=0}^{\infty} \|C_i f_i\|^2 = \frac{M^2}{l^2} \|f\|_C^2 < \infty$$

i.e. $f \in H^2(B)$.

For every $f \in H^2(C) = H^2(B)$ we have

$$\frac{l^2}{M^2} \|f\|_B^2 = \frac{l^2}{M^2} \sum_{i=0}^{\infty} \|B_i f_i\|^2 = \frac{l^2}{M^2} \sum_{i=0}^{\infty} \|B_i C_i^{-1} C_i f_i\|^2 \le$$

$$\le \frac{l^2 M^2}{M^2 l^2} \sum_{i=0}^{\infty} \|C_i f_i\|^2 = \|f\|_C^2 = \sum_{i=0}^{\infty} \|C_i f_i\|^2 = \sum_{i=0}^{\infty} \|C_i B_i^{-1} B_i f_i\|^2 \le$$

$$\leq \frac{L^2}{m^2} \sum_{i=0}^{\infty} \|B_i f_i\|^2 = \frac{L^2}{m^2} \|f\|_B^2$$

i.e.

$$\frac{l}{M}\|f\|_B \leq \|f\|_C \leq \frac{L}{m}\|f\|_B ,$$

which means that the norms $\|\cdot\|_B$ and $\|\cdot\|_C$ are equivalent. □

Corollary 1. *$H^2(B) = l^2(H)$ and the norms are equivalent.* □

Proposition 3. *Let $\{B_n\}$ and $\{C_n\}$ are sequences of positive, invertible, commuting operators on H with the property $0 < m < B_n < M$ and $0 < l < C_n < L$, then the operators U_+ on $H^2(B)$ and $H^2(C)$ are similar.*

Proof. The unilateral shifts U_+ on $H^2(B)$ and $H^2(C)$ are unitarily equivalent to operator valued weighted shifts $A' \sim (B_{n+1}B_n^{-1})$ and $A'' \sim (C_{n+1}C_n^{-1})$ respectively. We claim that the operators A' and A'' are similar. Indeed, let $X = [X_i]_{i=N}$ is a diagonal operator with diagonal elements:
$X_0 = 1,\ X_1 = B_1 B_0^{-1} C_0 C_1^{-1},\ X_2 = B_2 B_0^{-1} C_0 C_2^{-1},\ X_n = B_n B_0^{-1} C_0 C_n^{-1}, \cdots$
It is easy to verify that X is invertible operator and $AX = XB$.

So, the operators U_+ on $H^2(B)$ and $H^2(C)$ are similar. □

Proposition 4. *The unilateral shifts U_+ on $H^2(B)$ and $H^2(C)$ are unitarily equivalent if and only if there exists an unitary operator U on H such that the operator $B_n B_0^{-1} U C_0 C_n^{-1}$ is unitary for every $n \in N$.*

Proof. The operators U_+ on $H^2(B)$ and $H^2(C)$ are unitarily equivalent to operator valued weighted shifts $A' \sim (B_{n+1}B_n^{-1})$ and $A'' \sim (C_{n+1}C_n^{-1})$ respectively . By lemma 3.1 in [3], the operators A' and A'' are unitary equivalent if and only if there exists a unitary operator U on H such that for every $n \in N$ the operator $B_{n+1}B_n^{-1}B_n \cdots B_0^{-1} U C_0 C_1^{-1} C_1 \cdots C_n^{-1} C_n C_{n+1}^{-1}$ i.e. $B_n B_0^{-1} U C_0 C_n^{-1}$ is unitary on H. □

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УНИТАРНА ЕКВИВАЛЕНЦИЈА И СЛИЧНОСТ НА УНИЛАТЕЛАРНИ ШИФТОВИ НА ОПЕРАТОРСКО ТЕЖИНСКИ ПРОСТОРИ ОД НИЗИ

Резиме

Нека $\{B_i\}$е низа од позитивни, инверзибилни, комутирачки оператори на комплексен сепарабилен Хилбертов простор H. Просторот од вектори

$$H^2(B) = \left\{ f = (f_0, \ldots, f_n, \ldots) \colon f_i \in H,\ i = 0, 1, \ldots ; \sum_{i=0}^{\infty} \|B_i f_i\|^2 < \infty \right\}$$

е Хилбертов простор со скаларен производ

$$(f,g)_B = \sum_{I=0}^{\infty} (B_i f_i, B_i g_i)\,.$$

Во овој труд се дадени условите за сличност и унитарна еквиваленција на едностраните шифтови на просторите $H^2(B)$ и $H^2(C)$.

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