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ON OPERATOR VALUED WEIGHTED SHIFTS COMMUTING WITH U_+

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Abstract. In this paper we consider some properties of the operators K_{u+} and of the operators on K_A where $A \in K_{u+}$. Also we shall give some information about the operators that commutes with U_+. The computations show that operator $X \in B(\ell^2(H))$ which commutes with U_+ is the formal power series

$$U_+^* \overline{A_o} + \sum_{i=o}^{\infty} U_+^i \overline{A_{i+1}}, \quad \overline{A_i} \; K_{U_+}.$$

At the end we shall prove that if $(||A_i||)_{i=o}^{\infty} \in \ell^1$ then the formal power series is a bounded operator and is the limit in uniform operator topology of a sequence of polinomials in U_+.

Let H be a complex Hilbert space and $\ell^2(H) = \Sigma \oplus H_n$, $H_n = H$ for all n is the Hilbert space with scalar product defined by $(x,y) = \Sigma(x_n, y_n)$. Let $B(H)$ be the algebra of all (bound linear) operators from H to H, and let with B_{inv} we denote the set of all invertible operators on H.

One of the most interesting continous linear operators is the unilateral shift, the operator S defined on ℓ^2 by

$$S(x_o, x_1, \ldots) = (0, x_o, x_1, \ldots)$$

Coresponding to each sequence $\{w_n\}_{n=o}^{\infty}$ in ℓ^∞ there is the weighted unilateral shift W, defined on ℓ^2 by

$$W(x_o, x_1, \ldots) = (0, w_o x_o, w_1 x_1, \ldots)$$

Coresponding to each uniformly bounded sequence $(A_n)_{n=o}^{\infty}$ of bounded and linear operators on H, there is the unilateral operator valued weighted shift A defined on $\ell^2(H)$ by $A(x_o, x_1, \ldots) = (0, A_o x_o, A_1 x_1, \ldots)$ and we shall denote it by $A \sim (A_i)_{i=o}$.

If $A_n = 1$ for all n, then with U_+ we shall denote the operator valued weighted shift (OVWS) $U_+ \sim (1)$. With S we shall denote the set of all unilateral operator valued weighted shifts with invertible weights and $K_A = \{B \in S,\ AB=BA\}$, $A \in S$.

Throught this paper we consider some properties of the operators on K_{u+} and of the operators on K_A when $A \in K_{u+}$.

Proposition 1. If $B \sim (B_i) \in K_A$ where $A \sim (A_i)$, then $A_i \neq B_i$ for all i.

Proof. Suppose that there exists an $i_o > 0$ such that $A_{io}=B_{io}$ and $A_k \neq B_k$ for all $k < i_o$. Since $(A_{io-1}-B_{io-1})H \subset \mathrm{Ker} A_{io} = \{0\}$, this implies $A_{io-1}=B_{io-1}$. So $A_i = B_i$ for all $i \leq i_o$.

On the other hand $A_{io+1}B_{io} = B_{io+1}A_{io} = B_{io+1}B_{io}$, we have that $(A_{io+1}-B_{io+1})B_{io}H = \{0\}$ and so $A_{io+1} = B_{io+1}$ because $B_{io}H = H$. We get that $A_i = B_i$ for all $i \in \mathbb{N}$, i.e. A=B. ◊

Note that $A \in K_{u+}$ and $A \neq U_+$ then $A_i \neq 1$ for all i.

Proposition 2. $A \in K_{u+}$ iff $A_i = A_{i-1}$ for all i.

Proof. Let $A \in K_{u+}$, than $A_i 1 = 1A_{i-1}$ for all i. Suppose now that $A_i = A$ for all i and $\bar{A} \sim (A)$. Then $\bar{A}U_+ = U_+\bar{A}$ i.e. $\bar{A} \in K_{u+}$. ◊

Throught this paper $\bar{A} \sim (A)$ will be used to denote the operators on K_{u+}.

The question that arises is the connection between K_{u+} and B_{inv}. We shall prove that there exists an isometric isomorphism of K_{u+} onto B_{inv}. We shall need the following result:

Lemma 3. Let $A \sim (A_i)$ is the unilateral operator valued weighted shift. Then A is bounded iff $\sup_i ||A_i|| < \infty$ and then we have $||A|| = \sup_i ||A_i||$.

Proof. Let $f = (f_o, f_1, \ldots) \in \ell^2(H)$ and $M = \sup_i ||A_i|| < \infty$. $Af = (0, A_o f_o, A_1 f_1, \ldots)$ and so

$$||Af||^2 = \sum_{i=o}^{\infty} ||A_i f_i||^2 \leq \sum_{i=o}^{\infty} ||A_i||^2 ||f_i||^2 \leq M^2 ||f||^2.$$

It is then obvious that

$$||A|| = \sup_{f \neq o} \frac{||Af||}{||f||} \leq M.$$

Suppose now that $f_k^i \in H$ is such that $||f_k^i|| = 1$. Let $\tilde{f}_k = (0,0,\ldots,f_k^i,0,0,\ldots)$ where f_k^i is a vector on i position. Then we have $||\tilde{f}_k|| = 1$, $Af_k = (0,0,\ldots,A_i f_k^i,0,\ldots)$ and

$$||A|| \geq ||Af_k|| = ||A_i f_k^i||.$$

So $||A_i|| = \sup\limits_{||f_k^i||=1} ||A_i f_k^i|| \leq ||A||$ for all i and then

$$M = \sup_i ||A_i|| \leq ||A||. \quad \diamond$$

<u>Proposition 4</u>. K_{u+} is isometrically isomorphic to $\mathcal{B}_{inv}$.

<u>Proof</u>. Let A e $\mathcal{B}_{inv}$. We shall define the operator valued weighted shift $\bar{A} \sim (A)$, then $\bar{A} \in K_{u+}$. Let i: $\mathcal{B}_{inv} \to K_{u+}$ be defined by $i(A) = \bar{A}$. It is easy to verify that i is linear; by Proposition 2 i is surjective.

Also i is an isometry: $||i(A)||=||\bar{A}||=\sup||A_i||=||A||$. $\diamond$

<u>Proposition 5</u>. Let $\bar{A} \in K_{u+}$, then $B \sim (B_i) \in K_{\bar{A}}$ if and only if there exists $B_o \in \mathcal{B}_{inv}$ such that the sequence $(A^n B_o A^{-n})_n$ is uniformly bounded and $B_n = A^n B_o A^{-n}$.

<u>Proof</u>. Let $\bar{A} \in K_{u+}$; by Proposition 2 $\bar{A} \sim (A)$ if $B \sim (B_i) \in K_{\bar{A}}$, then $B_i A = AB_{i-1}$ for all i, i.e. $B_i = AB_{i-1}A^{-1} = A^i B_o A^{-i}$ and the sequence $(A^i B_o A^{-i})$ is uniformly bounded.

Conversly, let $B_o \in \mathcal{B}_{inv}$ be such that the sequence $(A^n B_o A^{-n})_n$ is uniformly bounded, and let $B_n = A^n B_o A^{-n}$. Then for the OVWS $B \sim (B_n)$ we have $B\bar{A} = \bar{A}B$ i.e. $B \in K_{\bar{A}}$. $\diamond$

Examples:

<u>1</u>. If $B_o = 1$, then $B = U_+$.

<u>2</u>. If $B_o = A$, then $B = \bar{A}$.

<u>3</u>. Let A be quasinormal (A commutates with A*A) and A = UP be the polar decomposition of A, then for $B_o = P$, $B = \bar{P} \in K_{u+} \cap K_{\bar{A}}$; and for $B_o = U$, $B = \bar{U} \in K_{u+} \cap K_{\bar{A}}$.

<u>4</u>. If B_o commutes with A then $B = \bar{B}_o \in K_{u+} \cap K_{\bar{A}}$.

<u>Proposition 6</u>. Let $\bar{A} \in K_{U+}$ and A = UP be the polar decomposition of A, then $\bar{A}$ is unitarily equivalent to an unilateral operator valued weighted shift with positive weights $Q \in K_{\bar{U}}$.

Proof. Let $\bar{A}\in K_{u+}$ and $A=UP$ be the polar decomposition of A. Then U is unitary operator and P is positive. Let $W = \mathrm{diag}(U^n)_{n=o}^{\infty}$ be the diagonal operator with the diagonal elements $(U^n)_{n=o}^{\infty}$, then $WU_+W^* = \bar{U}$, i.e. the operator valued weighted shift $\bar{U} \sim (U)$ is unitarily equivalent to U_+. Also $\bar{A} = \bar{U}\mathrm{diag}(P)$ is the polar decomposition of $\bar{A}$.

Then $\bar{A} = WU_+W^*\mathrm{diag}(P) = WU_+W^*\mathrm{diag}(P)WW^* = WQW^*$. Q is the operator valued weighted shift $U_+W^*\mathrm{diag}(P)W \sim (U^nPU^{-n})_{n=o}^{\infty}$ with positive weights. By Proposition 5, $Q\in K_{\bar{u}}$. ◊

Corollary 7. If A is quasinormal, then $\bar{A}$ is quasinormal and unitarily equivalent to $\bar{P}$.

Proof. A is quasinormal iff $PU = UP$. Then $Q = \bar{P}$. ◊

We can see now that if A and B are unitary operators on H, then $K_{\bar{A}}$ is isometrically isomorphic to $K_{\bar{B}}$.

Proposition 8. Let A be an unitary operator then $K_{\bar{A}}$ is isometrically isomorphic to B_{inv}.

Proof. Let $B_o\in B_{inv}$. We shall define the operator $B_1 = \mathrm{diag}(A^nB_oA^{-n})_{n=o}^{\infty}$ and $B = U_+B_1$.

B is bounded: $||B|| = \sup||A^nB_oA^{-n}|| \leq ||B_o|| < \infty$. So B is an operator valued weighted shift and $B\in K_{\bar{A}}$.

Let $i: B_{inv} \to K_{\bar{A}}$ be defined by $i(B_o) = B \sim (A^nB_oA^{-n})\in K_{\bar{A}}$. It is easy to verify that i is linear.

Let $C\in K_{\bar{A}}$. By Proposition 5, there exists $C_o\in B_{inv}$ such that $C \sim (A^nC_oA^{-n})_{n=o}^{\infty}$ i.e. $i(C_o) = C$. So, i is surjective. It remains to show that i is an isometry.

$$||i(B_o)|| = ||B|| = \sup_n ||A^nB_oA^{-n}|| = ||B_o|| \quad (\text{for } n=0). \quad ◊$$

Corollary 9. If A is an unitary operator, then there exists a map which is isometric isomorfism of K_{u+} onto $K_{\bar{A}}$.

Corollary 10. If A and B are unitary operators then $K_{\bar{A}}$ and $K_{\bar{B}}$ are isometrically isomorphic.

<u>Proof</u>. Let $i: K_{\overline{A}} \to K_{\overline{B}}$ be defined by $i: C \to C_1$ where $C \sim (A^n C_o A^{-n})_n$ and $C_1 \sim (B^n C_o B^{-n})_n$. Of course, i is isomorfism.

$||i(C)|| = ||C_1|| = ||C_o|| = ||C||$, so i is an isometry. ◊

<u>Proposition 11</u>. The operators A and $B \in B(H)$ are commuting iff $\overline{A} \in K_{\overline{B}}$.

<u>Proof</u>. Let AB = BA and $\overline{A} \sim (A)$. Then $A = A \cdot 1 = AB^n B^{-n}$ and so $\overline{A} \in K_{\overline{B}}$. Suppose now that $\overline{A} \in K_{\overline{B}}$. Then for some $A_o \in B_{inv} \overline{A} \sim (B^n A_o B^{-n})_n$, i.e. $A = B^n A_o B^{-n}$ for all n. For n=0 we get that $A_o = A$ and for $n = 1$, $A = BAB^{-1}$ i.e. AB = BA. ◊

Next we should like to give some information about the operators that commutates with U_+.

Let $(A_i)_{i=o}^{\infty}$ be a sequence of bounded linear invertible operators on H and $\overline{A}_i \sim (A_i) \in K_{u+}$.

A polynomial in U_+,

$$P_n(U_+) = U_+^* \overline{A} + \sum_{i=o}^{n-1} U_+^i \overline{A_{i+1}}, \quad \overline{A}_i \in K_{U_+} \qquad (*)$$

has the following matrix representation:

$$\begin{bmatrix} A_o & 0 & 0 & . \\ A_1 & A_o & 0 & . \\ . & . & . & . \\ A_n & A_{n-1} & A_{n-2} & . \\ 0 & A_n & A_{n-1} & . \\ 0 & 0 & A_n & . \\ . & . & . & . \end{bmatrix}$$

Note that the matrix is lower triangular and the non-zero en entries lie in a diagonal strip of width n+1. From (*) it is obvious that $||P_n(U_+)|| \le \sum_{i=o}^{n} ||A_i||$. So every polinomial of type (*) is bounded and commutes with U_+.

Now, let $X \in B(\ell^2(H))$ commutes with U_+. The computations show X is the formal power series

$$U_+^* \overline{A}_o + \sum_{i=o}^{\infty} U_+^i \overline{A_{i+1}}, \quad \overline{A}_i \in K_{u+} \qquad (**)$$

X has the following matrix:

$$\begin{bmatrix} A_o & 0 & 0 & . \\ A_1 & A_o & 0 & . \\ . & . & . & . \\ A_n & A_{n-1} & A_{n-2} & . \\ A_{n+1} & A_n & A_{n-1} & . \\ A_{n+2} & A_{n+1} & A_n & . \\ . & . & . & . \end{bmatrix}$$

Note that if X is bounded, then (A_i) is uniformly bounded sequence of operators. Let $f_o \in H$, $||f_o|| = 1$ and $f = (f_o, 0, 0, \ldots)$.

Then $||f|| = 1$ and $\infty > ||X||^2 \geq ||Xf||^2 = \sum_{i=o}^{\infty} ||A_i f_o|| \geq$ $\geq ||A_i f_o||^2$ for all i, and for all $f_o \in H$, $||f_o|| = 1$.

Then $||A_i|| = \sup_{||f_o||=1} ||A_i f_o|| \leq ||X|| < \infty$, for all $i \Rightarrow \sup_i ||A_i|| \leq ||X|| < \infty$.

Note also that if the sequence (A_i) is uniformly bounded, then X is not neceserily bounded on $\ell^2(H)$. For example, let A_i be unitary for all i and let $X = U_+^* \bar{A}_o + U_+^i \overline{A_{i+1}}$. Let $f_o \in H$ be such that $||f_o|| = 1$ and $f = (f_o, 0, 0, \ldots)$. Then

$$||X||^2 \geq ||Xf||^2 = \sum_{i=o}^{\infty} ||A_i f\ ||^2 = \sum_{i=o}^{\infty} ||f_o||^2 = \infty.$$

The natural question that arises from the above discusion is the following:

When the formal power series (**) defines a bounded operator?

We shall prove the following:

<u>Proposition 12</u>. If $\{||A_i||\}_{i=o}^{\infty} \in \ell^1$, then the formal power series (**) is a bounded operator and is the limit, in uniform operator topology of a sequence of polinomials in U_+.

<u>Proof</u>. Let $\{||A_i||\}_{i=o}^{\infty} \in \ell^1$ and $X = U_+^* \bar{A}_o + \sum_{i=o}^{\infty} U_+^i \overline{A_{i+1}}$, $\bar{A}_i \sim (A_i) \in K_{u+}$.

Let $\varepsilon > 0$, then there exists $n_o \in \mathbb{N}$ such that for $n \geq n_o$, $\sum_{i=n+1}^{\infty} ||A_i|| < \sqrt{\varepsilon}$.

Let $f = (f_o, f_1, \ldots) \in \ell^2(H)$ is such that $||f|| = 1$.

Then:

$$(X-P_n(U_+))f = (0,0,A_{n+1}f_o,A_{n+2}f_o+A_{n+1}f_1,$$

$$A_{n+3}f_o+A_{n+2}f_1+A_{n+1}f_2,\ldots,\sum_{k=o}^{m} A_{n+m-k+1}f_k,\ldots)$$

and so:

$$||(X-P_n(U_+)f||^2=||A_{n+1}f_o||^2+||A_{n+2}f_o+A_{n+1}f_1||^2+$$

$$||A_{n+3}f_o+A_{n+2}f_1+A_{n+1}f_2||^2+\ldots \le$$

$$||A_{n+1}f_o||^2+||A_{n+2}f_o||^2+||A_{n+1}f_1||^2+2||A_{n+2}f_o||\ ||A_{n+1}f_1|| +$$

$$||A_{n+3}f_o||^2+||A_{n+2}f_1||^2+||A_{n+1}f_2||^2+2||A_{n+3}f_o||\ ||A_{n+2}f_1|| +$$

$$2||A_{n+3}f_o||\ ||A_{n+1}f_2||+2||A_{n+2}f_1||\ ||A_{n+1}f_2||+\ldots =$$

$$\sum_{i=o}^{\infty}||A_{n+1}f_i||^2 + \sum_{i=o}^{\infty}||A_{n+2}f_i||^2+\ldots+2\sum_{i=o}^{\infty}||A_{n+2}f_i||\ ||A_{n+1}f_{i+1}|| +$$

$$2\sum_{i=o}^{\infty}||A_{n+3}f_i||\ \ ||A_{n+1}f_{i+2}||+\ldots <||A_{n+1}||^2||f||^2+||A_{n+2}||^2||f||^2+\ldots+$$

$$2(\sum_{i=o}^{\infty}||A_{n+2}f_i||^2)^{1/2}(\sum_{i=o}^{\infty}||A_{n+1}f_i||^2)^{1/2} +$$

$$2(\sum_{i=o}^{\infty}||A_{n+3}f_i||^2)^{1/2}(\sum_{i=o}^{\infty}||A_{n+1}f_{i+2}||^2)^{1/2} +\ldots \le$$

$$\le \sum_{i=o}^{\infty}||A_{n+i}||^2+2||A_{n+2}||\ ||A_{n+1}||+2||A_{n+3}||\ ||A_{n+1}|| +\ldots =$$

$$(\sum_{i=n+1}^{\infty}||A_i||)^2 < \varepsilon.$$

$$\Rightarrow ||X-P_n(U_n(U_+)|| = \sup_{||f||=1}||(X-P_n(U_+)).f|| \le \varepsilon,$$

i.e. X is the limit in uniform operator topology of a sequence of polinomials in U_+.

Now from the inequality:

$$\Big|\,||X|| - ||P_n(U_+)||\,\Big| \le ||X-P_n(U_+)|| < \varepsilon$$

we get that

$$||P_n(U_+)||-\varepsilon \le ||X|| \le||P_n(U_+)||+\varepsilon \le \sum_{i=o}^{\infty}||A_i|| + \varepsilon < \infty$$

i.e. X is a bounded operator. ◊

R E F E R E N C E S

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ЗА ОПЕРАТОРСКО ТЕЖИНСКИТЕ ШИФТОВИ ШТО КОМУТИРААТ СО U_+

Марија Оровчанец

Р е з и м е

Во овој труд разгледуваме некои својства на оператори од K_{u+} и од K_A каде што $A \in K_{u+}$. Исто така даваме некои информации за оператори што комутираат со U_+. Пресметувањата покажуваат дека операторот $X \in \mathcal{B}(\ell^2(H))$ што комутира со U_+ е формалниот степенски ред

$$U_+^* \overline{A}_o + \sum_{i=o}^{\infty} U_+^i \overline{A_{i+1}}, \quad \overline{A}_i \in K_{U_+}.$$

На крај покажуваме дека ако низата $(||A_i||)_{i=o}^{\infty} \in \ell^1$ тогаш формалниот степенски ред е ограничен оператор и е граница, во рамномерна оператор топологија, на низа од полиноми од U_+.

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