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CONNECTEDNESS OF INVERSE LIMIT OF GENERALIZED TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to study the connectednes of inverse limit of generalized topological spaces introduced by Császár in [3].

1. Introduction

This section contains some basic definitions and propositions concerning supra topological and generalized topological spaces.

Definition 1. Let J be any nonempty indexed set and let X be a nonempty set. A subfamily μ of X is said to be *supra topology* on X if:

i) $X, \emptyset \in \mu$
ii) If $A_i \in \mu$ for all $i \in J$, then $\cup A_i \in \mu$.

This definition can be reformulated as follows.

Definition 2. A subcollection $\mu \subset 2^X$ is called a *supra topology* on X, [6], if $X \in \mu$ and μ is closed under arbitrary union.

Definition 3. [10, Definition 2.3] Let $(X,\ \tau)$ be a topological space and μ be a supra topology on X. We call μ a supra topology *associated* with τ if $\tau \subseteq \mu$.

Proposition 1.1. [11, Theorem 1] *If μ is a supra topology on X, then $T_\mu = \{A \subset X : A \cap B \in \mu$ for each $B \in \mu\}$ is a topology and $T_\mu \subset \mu$.*

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Definition 4. (X, μ) is called a *supra* topological space. The elements of μ are said to be *supra open* in (X, μ) and the complement of a supra open set is called a *supra closed* set.

Császár in [3] introduced the notion of generalized topological space (GTS) as another name for supra topology. He also introduced the notion of (μ_1, μ_2)-continuous function on GTS's and the separation axioms defined by replacing open sets by members of a GTS.

In general, let M_μ denote the union of all elements of μ; of course, $M_\mu \in \mu$, and $M_\mu = X$ if and only if μ is a strong general topology (strong GT). A subset A of X is called μ-open if $A \in \mu$. A subset B of X is called μ-closed if $X \backslash B \in \mu$.

A point $x \in X$ is called a *μ-cluster point of* A if $U \cap (A \backslash \{x\}) \neq \emptyset$ for each U in μ such that $x \in U$.

Let $\mathcal{B} \subset expX$ satisfy $\emptyset \in \mathcal{B}$. Then all unions of some elements of $\mathcal{B}$ constitute a GT $\mu(\mathcal{B})$, and $\mathcal{B}$ is said to be a *base* for $\mu(\mathcal{B})$.

Definition 5. [12, Definition 2.1] Let X be a space. Then $\mu_x = \{U : x \in U \in \mu\}$.

Definition 6. [12, Definition 2.2] Let (X, μ) be a GTS. X is called a μT_2-space if X satisfies the following μT_2-separation conditions: If $x, y \in X$ and $x \neq y$, then there are $U_x \in \mu_x$ and $U_y \in \mu_y$ such that $U_x \cap U_y = \emptyset$.

Definition 7. Let (X, μ) be a generalized topological space. Then X is called a *μT_1-space* if for $x_1, x_2 \in M_\mu$ with $x_1 \neq x_2$, there exist $U, V \in \mu$ such that $x_1 \in U, x_2 \notin U$ and $x_2 \in V, x_1 \notin V$.

Definition 8. Let μ be a GT on X. We say that $M \subset X$ is *μ-open* if and only if $M \in \mu$; $N \subset X$ is *μ-closed* if and only if $X - N \in \mu$.

Definition 9. ([7]) If $A \subset X$ then $i_\mu A$ denotes the union of all μ-open sets contained in A and $c_\mu A$ is the intersection of all μ-closed sets containing A.

Both i_μ and c_μ are idempotent operations (where the operation γ is said to be *idempotent* if and only if $\gamma\gamma A = \gamma A$ for $A \subset X$).

Proposition 1.2. *For $A \subset X$ and $x \in X$, we have $x \in c_\mu A$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$.*

Definition 10. Let μ be a GT on X, μ' a GT on X' and $f : X \to X'$. We say that the map f is $(\mu,\ \mu')$-*continuous* if and only if $M' \in \mu'$ implies $f^{-1}(M') \in \mu$, and $(\mu,\ \mu')$-*open* if and only if $M \in \mu$ implies $f(M) \in \mu'$. If f is bijective and $(\mu,\ \mu')$-continuous, moreover f^{-1} is $(\mu',\ \mu)$-continuous, then it is natural to say that f is a $(\mu,\ \mu')$-*homeomorphism*.

Let X be a non-empty set and let $\mathcal{B}$ be a collection of subsets of X with $\emptyset \in \mathcal{B}$. Then the collection of all possible unions of elements of $\mathcal{B}$ forms a GT $\mu(\mathcal{B})$ on X and $\mathcal{B}$ is called a base for $\mu(\mathcal{B})$.

Now, let $A \neq \emptyset$ be an index set, $X_a \neq \emptyset$ for $a \in A$, and $X = \prod\{X_a : a \in A\}$ the Cartesian product of the sets X_a. We denote by p_a the *projection* p_a: $X \to X_a$.

Definition 11. ([4]) Suppose that, for $a \in A$, μ_a is a given GT on X_a. Let us consider all sets of the form $\prod\{M_a : M_a \in \mu_a\}$ and, with the exception of a finite number of indices a, $M_a = X_a$. We denote by $\mathfrak{B}$ the collection of all these sets. Clearly $\emptyset \in \mathfrak{B}$ so that we can define a GT $\mu = \mu(\mathfrak{B})$ having $\mathfrak{B}$ for base. We call μ the *product* of the GT s μ_a and denote it by $P_a\mu_a$.

The base for $\prod\{X_a : a \in A\}$ described in above Definition is called the *canonical base* for the Cartesian product. If each μ_a is a topology then clearly μ is the product topology of the factors μ_a.

Proposition 1.3. *If $B \in \mathfrak{B}$, then there exist a finite number of indices $a_1, a_2, ..., a_n$ such that $B = p_{a_1}^{-1}(M_{a_1}) \cap\ p_{a_2}^{-1}(M_{a_2}) \cap\ ... \cap\ p_{a_2}^{-1}(M_{a_2})$.*

Let us write $i = i_\mu,\ c = c_\mu,\ i_k = i_{\mu_k},\ c_k = c_{\mu_k}$.

Consider in the following $A_k \subset X_k,\ A = \prod_{k \in K} A_k,\ x \in X$ and $x_k = p_k(x)$.

Proposition 1.4. [4, Proposition 2.1.] $iA \subset \prod_{k \in K} i_k A_k$.

Proof. If $x \in iA$ then there is $M \in \mu$ such that $x \in M \subset A$. Then there are sets $M_k \in \mu_k$ such that $x \in \prod_{k \in K} M_k \subset M \subset A$. For $p_k(x) = x_k$, we have $x_k \in M_k$ so that $M_k \neq \emptyset$ and therefore $\prod_{k \in K} M_k \subset \prod_{k \in K} A_k$ implies $M_k \subset A_k$ for each k. Thus $x_k \in M_k \subset A_k$ shows that $x_k \in i_k A_k$. □

Similarly, we have the following result.

Proposition 1.5. [3, Proposition 2.3.] $cA = \prod_{k \in K} c_k A_k$.

Proposition 1.6. [4, Proposition 2.4] *The projection p_k is $(\mu,\ \mu_k)$-open.*

In general, p_k need not be $(\mu,\ \mu_k)$ continuous.

Proposition 1.7. [4, Proposition 2.7] *If every μ_k is strong then μ is strong and p_k is $(\mu,\ \mu_k)$- continuous for $k \in K$.*

2. Inverse systems and limits

Let A be a set directed by the relation $\leq$ and let $\{X_a : a \in A\}$ be a family of sets indexed by A. For each pair (a, b) of elements of A such that $a \leq b$, let p_{ab} be a mapping of X_b into X_a, i.e. $p_{ab} : X_b \to X_a$. Suppose that p_{ab} satisfy the following conditions [2, p. 191]:

($\mathbf{LP}_{\mathrm{I}}$): The relations $a \leq b \leq c$ imply $p_{ac} = p_{ab}p_{bc}$,

($\mathbf{LP}_{\mathrm{II}}$): For each $a \in A$, p_{aa} is the identity mapping of X_a.

Then we say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an *inverse system* of sets X_a and *bonding* mapping p_{ab}.

Let $Y = \Pi\{X_a : a \in A\}$ be the product of the family of sets $\{X_a : a \in A\}$ and X denote the subset of Y consisting of all $x = (x_a : a \in A)$ (called a *thread* of $\mathbf{X}$) which satisfy each of the relation $x_a = p_{ab}(x_b)$ for each pair of indices (a, b) such that $a \leq b$. The set X is called inverse limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$.

Proposition 2.1. [5, 2.5.1. Proposition] *The limit of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of Hausdorff spaces X_a is the closed subset of the Cartesian product $\Pi\{X_a : a \in A\}$.*

This Proposition is true for inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ of generalized spaces.

Proposition 2.2. *The limit of an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of μT_2 generalized topological spaces X_a is the closed subset of the Cartesian product $\Pi\{X_a : a \in A\}$.*

Proof. Let us prove that each point $x = \{x_a : a \in A\} \notin \lim \mathbf{X}$ has a neighbourhood which is disjoint with $\lim \mathbf{X}$. From $\{x_a : a \in A\} \notin \lim \mathbf{X}$ it follows that there exist a $b \in A$ such that for $c \leq b$ we have $p_{bc}(x_b) \neq x_c$. There exist disjoint $\mu-$open sets U_c and V_c such that $x_c \in U_c$ and $p_{bc}(x_b) \in V_c$. Now $U_b = p_{bc}^{-1}(U_c) \cap p_{bc}^{-1}(V_c)$ is $\mu-$open set containing x_b. It follows that $U = p_b^{-1}(U_b)$ is an $\mu-$open set containing $x = \{x_a : a \in A\}$. Let us prove $U \cap \lim \mathbf{X} = \emptyset$. If $y = \{y_a : a \in A\} \in U$ then $p_b(y) \in U_b = p_{bc}^{-1}(U_c) \cap p_{bc}^{-1}(V_c)$.

It follows that $p_{bc}(y_b) \neq y_c$. Thus, $y \notin \lim \mathbf{X}$. This means that $U \cap \lim \mathbf{X} = \emptyset$ and the proof is completed. □

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of generalized topological spaces and let $X = \lim \mathbf{X}$. For every $a \in A$ there is a continuous mapping $p_a = P_a|X : X \to X_a$, where $P_a : \Pi\{X_a : a \in A\} \to X_a$ is the projection,

Definition 12. Let $U \in \mu$ so that $x \in U$ and U is contained in every $V \in \mu$ with $x \in V$. In this case we call x a representative element for U.

Definition 13. A space X that every $x \in X$ is a representative element for some $U \in \mu_x$ is called a C_0-space in [12].

Proposition 2.3. *The family of all sets $p_a^{-1}(U_a)$, where U_a is an $\mu-$open subset of X_a and runs over a subset A^l cofinal in A, is a base for the limit of the inverse system $\mathbf{X}$ =$\{X_a, p_{ab}, A\}$. Moreover, if for every $a \in A$ base B_a for X_a is fixed, then the subfamily consisting of those $p_a^{-1}(U_a)$ in which $U_a \in B_a$, also is a base.*

3. Quasi-compact generalized topological spaces

Definition 14. A generalized topological space (X, μ) is $\mu-$*quasi-compact* if every cover of $\mu-$ *open* subsets of (X, μ) has the finite subcover.

Theorem 1. *A generalized topological space (X, μ) is $\mu-$quasi-compact if and only if every family of $\mu-$closed subsets of (X, μ) which has the finite intersection property has non-empty intersection.*

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system such that for every $a \in A$ there exists a family $\mathcal{S}_a$ of subsets of X_a with the following properties, [2, p. 190]:

- (**I**) Arbitrary intersection of the sets from $\mathcal{S}_a$ is a set from $\mathcal{S}_a$,
- (**II**) If a family of subsets $\mathcal{F} \subset \mathcal{S}_a$ has the finite intersection property, then $\cap\{M : M \in \mathcal{F}\}$ is non-empty,
- (**III**) $p_{ab}^{-1}(x_a) \in \mathcal{S}_b$ for every $x_a \in X_a$ and for every pair a, b, $a \leq b$,
- (**IV**) $p_{ab}(M_b) \in \mathcal{S}_a$ for every $M_b \in \mathcal{S}_b$ and for every pair a, b, $a \leq b$.

Theorem 2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system which satisfies conditions* (**I**), (**II**), (**III**) *and* (**IV**). *If $X = \lim \mathbf{X}$, then:*

a) *For all $a \in A$*

$$p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\}, a \in A, \tag{3.1}$$

b) *If* $X_a \neq \emptyset$ *for every* $a \in A$, *then* $X \neq \emptyset$.

Proof. Let S_a be a family of all non-empty subsets of X_a and let $\mathcal{Y}$ be a family of all collections $Y = \{Y_a : Y_a \in S_a,\ a \in A\}$ such that $p_{ab}(Y_b) \subset Y_a$. The family $\mathcal{Y}$ is non-empty since $\mathbf{X} \in \mathcal{Y}$. For two collections $Y = \{Y_a : Y_a \in S_a,\ a \in A\}$ and $Z = \{Z_a : Z_a \in S_a, a \in A\}$ we shall write $Y \geq Z$ if $Y_a \subset Z_a$ for every $a \in A$. It is clear that $(\mathcal{Y},\ \geq)$ is a partially ordered set. The remaining part of the proof consists of several steps.

Step 1. *There exists a maximal element in* $(\mathcal{Y},\ \geq)$.

It suffices to prove that $(\mathcal{Y},\ \geq)$ is inductive, i.e. if $L = \{Y^\lambda : \lambda \in \Lambda\}$ is a strictly increasing chain in $(\mathcal{Y},\ \geq)$, then there is an element $M \in (\mathcal{Y},\ \geq)$ such that $M \geq Y^\lambda$ for every $\lambda \in \Lambda$. We define $M = \{M_a : M_a \in S_a,\ a \in A\}$ such that $M_a = \cap\{Y_a^\lambda : \lambda \in \Lambda\}$. From the properties (I) and (II) it follows that the set M_a is non-empty $\mathcal{S}_a$ subset of X_a. Moreover, $p_{ab}(M_b) \subset M_a$.

Step 2. *If* $Y = \{Y_a : Y_a \in \mathcal{S}_a,\ a \in A\}$ *is a maximal element of* $(\mathcal{Y},\ \geq)$, *then* $Y_a = p_{ab}(Y_b)$ *for every pair* $a,\ b \in A$ *such that* $a \leq b$.

Let $Z = \{Z_a : Z_a \in \mathcal{S}_a, a \in A\}$ be a collection such that $Z_a = \cap\{p_{ab}(Y_b) : b \geq a\}$. Each $p_{ab}(Y_b) \in \mathcal{S}_a$ since $Y_b \in \mathcal{S}_b$. From the properties (I) and (II) it follows that the set Z_a is non-empty S_a subset of X_a. In order to prove that $Z \in (\mathcal{Y},\ \geq)$ it suffices to prove that $p_{ab}(M_b) \subset M_a$. If $a \leq b$ then $p_{ab}(Z_b) \subset \cap\{p_{ab}(p_{bc}(Y_c)) : b \leq c\} = \cap\{p_{ac}(Y_c) : c \geq b\}$. On the other hand, for every $d \geq a$ there is a $c \in A$ such that $c \geq b,\ d$. It follows that $p_{ac}(Y_c) \subset p_{ad}(Y_d)$. This means that

$$\cap\{p_{ac}(Y_c) : c \geq b\} = \cap\{p_{ad}(Y_d) : c \geq b\} = Z_a.$$

Finally, we have $Z \in (\mathcal{Y},\ \geq)$. Moreover, $Z_a \subset Y_a$ for each $a \in A$. This means that $Z = Y$ since Y is maximal.

Step 3. *If* $Y = \{Y_a : Y_a \in \mathcal{S}_a, a \in A\}$ *is a maximal element of* $(\mathcal{Y}, \geq)$, *then* Y_a *is one-point set for every* $a \in A$.

Let $x_a \in Y_a$. Define

$$Z_b = \begin{cases} Y_b \cap p_{ab}^{-1}(x_a), & \text{if} \quad b \geq a, \\ Y_b, & \text{if} \quad b \not\leq a. \end{cases}$$

Let us prove that $Z = \{Z_a : Z_a \in \mathcal{S}_a, a \in A\}$. We infer that each $Y_b \cap p_{ab}^{-1}(x_a)$ is in $\mathcal{S}_a$. It is easy to prove that $p_{ab}(Z_b) \subset Z_a$. Hence, $Z \in (\mathcal{Y}, \geq)$. Now, $Z = Y$ since $Z \geq Y$ and Y is maximal. This means $Y_a = \{x_a\}$.

Step 4. $\lim \mathbf{X}$ *is non-empty.*

From Step 3 we have that $Z = \{Z_a : Z_a \in \mathcal{S}_a,\ a \in A\} = \{x_a : a \in A\}$ such that $p_{ab}(x_b) = x_a$ for every pair $a,\ b$ such that $b \geq a$.

Step 5. *Let us prove that* $p_a(X) = \cap \{p_{ab}(X_b) :\ b \geq a\}$.

It is clear that $p_a(X) \subset \cap\{p_{ab}(X_b) : b \geq a\}$. Let us prove that $p_a(X) \supset \cap\{p_{ab}(X_b) : b \geq a\}$. Let $x_a \in \cap\{p_{ab}(X_b) : b \geq a\}$. This means that $Y_b = p_{ab}^{-1}(x_a)$ is non-empty for each $b \geq a$. Moreover, $Y_b \in \mathcal{S}_b$. For each b non-comparable with a, let $Y_b = X_b$. Now, we have a collection $Y = \{Y_a : Y_a \in \mathcal{S}_a,\ a \in A\}$ which is evidently in $(\mathcal{Y},\ \geq)$. There exists a maximal element $Z = \{Z_a : Z_a \in \mathcal{S}_a,\ a \in A\}$ in $(\mathcal{Y},\ \geq)$ such that $Z \geq Y$. It follows that each Y_a is some Z_a which is a point $z_a \in X_a$ (Step 3) since Z is maximal. The collections (z_a) is a point of $\lim \mathbf{X}$. Hence, $p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\}$. □

Theorem 3. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of non-empty* $\mu-$*quasi-compact* μT_1 *spaces* X_a *and* $\mu-$*closed mappings* p_{ab}. *Then* $\lim \mathbf{X}$ *is non-empty.*

Proof. Let $\mathcal{S}_a$ be a family of $\mu-$closed subsets of X_a and let each p_{ab} be a closed mapping. Now, we have the following properties:

(I) arbitrary intersection of μ-closed sets from $\mathcal{S}_a$ is a $\mu-$closed set from $\mathcal{S}_a$, since arbitrary union of the μ-open sets is a μ-open.

(II) If a family of subsets $\mathcal{F} \subset \mathcal{S}_a$ has the finite intersection property, then $\cap\{M : M \in \mathcal{F}\}$ is non-empty since each X_a is $\mu-$quasi-compact.

(III) $p_{ab}^{-1}(x_a) \in \mathcal{S}_b$ for every $x_a \in X_a$ and for every pair $a, b,\ a \leq b$, since in T_1 space each point is closed.Thus, is $p_{ab}^{-1}(x_a)$ $\mu-$closed.

(IV) $p_{ab}(M_b) \in \mathcal{S}_a$ for every $M_b \in \mathcal{S}_b$ and for every pair $a, b,\ a \leq b$ since p_{ab} is μ-closed.

Using Theorem 2 we complete the proof. □

Remark 3.1: In fact, from the proof of Theorem 2 it follows that each closed subsystem $\mathbf{Z}$ contains some minimal closed subsystem $\mathbf{Y}$.

Now we prove the quasi-compactness of the limit space.

Theorem 4. *Let* $\mathbf{X} = \{X_\alpha,\ p_{\alpha\theta},\ A\}$ *be an inverse system of quasi- compact* μT_1 *spaces* X_α *and* $\mu-$*closed mappings* $p_{\alpha\beta}$. *Then* $\lim \mathbf{X}$ *is* $\mu-$*quasi-compact.*

Proof. Let $\mathcal{U} = \{U_\mu \ : \ \mu \in M\}$ be an $\mu-$open cover of $\lim \mathbf{X}$. By virtue of the definition of a base in $\lim \mathbf{X}$ there is an open set $U_{\mu,a} \subseteq X_\alpha$, for each $\alpha \in A$ and $\mu \in M$, such that $U_\mu = \Pi\{U_{\mu,\alpha} : \alpha \in A\}, p_\alpha^{-1}(U_{\mu,\alpha}) \subseteq U_\mu$ and $U_{\mu,\alpha}$ is a maximal set with respect to property $p^{-1}(U_{\mu,\alpha}) \subseteq U_\mu$. Let $\mathcal{U}_\alpha$ be a family $\{U_{\mu,\alpha\backslash}. \ \alpha \in A\}$. If $\mathcal{U}_\alpha$ is the cover of X_a then $p_\alpha^{-1}(\mathcal{U}_\alpha)$ is a cover $\lim \mathbf{X}$ which refines $\mathcal{U}$. This means that $\mathcal{U}$ has a finite subcover since $\mathcal{U}_\alpha$ has a finite subcover. Now we prove that there exists an $\alpha \in A$ such $\mathcal{U}_\alpha$ is a cover of X_α. In the opposite case the set $Z_\alpha = X_\alpha \backslash (\cup\{U_{\mu\alpha} : \mu \in M\}\rangle$ is non-empty for each $\alpha \in A$. Now we obtain a closed subsystem $\mathbf{Z} = \{Z_\alpha, , \ p_{a\beta}|Z_\beta, \ A\}$. By virtue of Remark 3.1 it follows that there is a closed subsystem $\mathbf{Y} \leq \mathbf{Z}$ such that $\mathbf{Y}$ is minimal. From the proof of Theorem 2 it follows that $\lim \mathbf{Y}$ is non-empty. This means that $\lim \mathbf{Z} \neq \emptyset$. Let z be any point of $\lim \mathbf{Z}$. It is easy to prove that $z \notin \cup\{f_\alpha^{\to 1}(U_{\mu,\alpha}) \ : \ \alpha \in A, \ \mu \in M\}$. This is impossible since $\mathcal{U} = \{U_\mu \ : \ \mu \ \in \ M\}$ is the cover of $\lim \mathbf{X}$. Thus, there exists an $\alpha \in A$ such $\mathcal{U}_\alpha$ is a cover of X_α. The proof is complete. □

4. Connectedness of inverse limit of generalized connected spaces

Definition 15. [1, Definition 3.1] Let $(X, \ \mu)$ be a GTS. X is called μ-*connected* if there are no nonempty disjoint μ-open subsets U, V of X such that $U \cup V = X$.

Theorem 5. *A generalized topological space (X, μ) is $\mu-$quasi-compact if and only if every family of $\mu-$closed subsets of (X, μ) which has the finite intersection property has non-empty intersection.*

Let us recall, [1], that if $(X, \ \mu)$ GTS then X is called μ-connected if there are no nonempty disjoint μ-open subsets U, V of X such that $U \cup V = X$.

Corollary 5.1. *It is clear that X is μ-connected if and only if there are no nonempty disjoint μ-closed subsets F_1, F_2 of X such that $F_1 \cup F_2 = X$.*

We shall prove the following result.

Theorem 6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of $\mu-$quasi-compact spaces X_a such that $X = \lim \mathbf{X}$ is $\mu-$quasi-compact and surjective projections $p_a : X \to X_a$. Then $\lim \mathbf{X}$ is $\mu-$connected if each X_a is $\mu-$connected.*

Proof. Suppose that X is not $\mu-$connected. It follows that there exist μ-closed sets F_1, F_2 such that

$$F_1 \cup F_2 = X \tag{1}$$

and

$$F_1 \cap F_2 = \emptyset. \tag{2}$$

Consider the sets

$$Y_a = c_\mu(p_a(F_1)) \cap c_\mu(p_a(F_2)) \tag{3}$$

From the ontoness of $p_a : X \to X_a$ it follows that

$$c_\mu(p_a(F_1)) \cup c_\mu(p_a(F_2)) = X_a. \tag{4}$$

This relations implies that $Y_a \neq \emptyset$ since from $Y_a = \emptyset$ it follows that X_a is not μ-connected.

Now the family $\{p_a^{-1}(Y_a) : a \in A\}$ is the family with finite intersection property of μ-closed subset of μ-quasi-compact space $X = \lim \mathbf{X}$. Thus

$$Y = \cap\{p_a^{-1}(Y_a) : a \in A\} \neq \emptyset \tag{5}$$

Let $y \in Y$. From the relations (1) and (2) that $y \in F_1/F_2$ or $y \in F_2/F_1$. In any case there exists $b \in A$ and a μ-open set $V_b \subset X_b$ such that $p_b^{-1}(V_b) \cap F_2 = \emptyset$ or $p_b^{-1}(V_b) \cap F_1 = \emptyset$. It follows that $p_b(y) \notin p_b(F_2)$ or $p_b(y) \notin p_b(F_1)$. We infer that $p_b(y) \notin Y_b$. This is impossible because (5). This means that $F_1 = \emptyset$ or $F_2 = \emptyset$, i.e. that X is $\mu-$connected. □

5. Connectedness if inverse limit of weakly μ-compact spaces

Definition 16. A function $f : (X,\ \mu) \to (Y,\ \kappa)$ is called $(\mu,\ \kappa)-$*continuous* if the inverse image of each κ-open set is μ-open.

Definition 17. Let A be a nonempty subset of a space $(X;\ \mu)$. The generalized subspace topology on A is the collection $\{U \cap A : U \in \mu\}$, and will be denoted by μ_A. The generalized subspace A is the generalized topological space $(A,\ \mu_A)$.

Definition 18. [8, Definition 2.1.] A space $(X;\ \mu)$ is called *weakly μ-compact* (briefly *$w\mu-$compact*) if any μ-open cover of X has a finite subfamily, the union of the μ-closures of whose members covers X.

It is clear that every μ-compact space $(X;\ \mu)$ is $w\mu$-compact.

Lemma 1. [8, Proposition 2.7.] *A space $(X,\ \mu)$ is $w\mu$-compact if and only if any μ-regular open cover of X has a finite subfamily, the union of the $\mu-$ closures of whose members covers X.*

Lemma 2. [8, Proposition 2.8.] *For a space $(X;\ \mu)$, the following are equivalent:*

(i) *X is $w\mu$-compact,*
(ii) *For any family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\cap\{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\cap\{i_\mu(U_\alpha) : \alpha \in \Lambda_0\} = \emptyset$,*
(iii) *For any family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of μ-regular closed subsets of X such that $\cap\{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\cap\{i_\mu(U_\alpha) : \alpha \in \Lambda_0\} = \emptyset$.*

Definition 19. [8, Definition 2.9.] Let A be a subset of a space $(X,\ \mu)$. A point $x \in X$ is said to be a *θ_μ-accumulation point of A* if $c_\mu(U) \cap A \neq \emptyset$ for every μ-open subset U of X that contains x. The set of all θ_μ-accumulation points of A is called the *θ_μ-closure of A* and is denoted by $(c_\mu)_\theta(A)$. A is said to be *μ_θ-closed* if $(c_\mu)_\theta(A) = A$. The complement of a μ_θ-closed set is called *μ_θ-open*.

It is clear that A is μ_θ-open if and only if for each $x \in A$, there exists a μ-open set U such that $x \in U \subset c_\mu(U) \subset A$.

Theorem 7. *If X is a μ-space then:*

(a) *the empty set and the whole space are μ_θ-closed,*
(b) *arbitrary intersection of μ_θ-closed sets are μ_θ-closed,*
(c) *$c_\mu(K) \subset (c_\mu)_\theta(K)$ for each subset K,*
(d) *a μ_θ-closed subset is closed.*

Lemma 3. *If X is a μ- space, then for each $Y \subset X$ there exists a minimal μ_θ-closed subset $Z \subset X$ such that $Y \subset Z$.*

Proof. The collection Φ of all μ_θ-closed subsets W of X which contains Y is non-empty since $X \in \Phi$. By (b) of Theorem 7 we infer that $Z = \cap\{W : W \in \Phi\}$ is a minimal μ_θ-closed subset $Z \subset X$ containing Y. □

From Theorem 7 it follows that the family of all μ_θ-open subsets of (X, μ) is a new generalized topology (GT) μ_θ on X.

Definition 20. Let (X, μ) be a GT space. The μ_θ-space of X is the space (X, μ_θ).

In the sequel we shall use denotations X_μ and X_{μ_θ}.

Definition 21. Let X be a space. Then $\mu_x = \{U : x \in U \in \mu\}$.

Lemma 4. *If X_μ is a μT_2 space, then X_{μ_θ} is μT_1-space.*

Proof. Let x be any point of X_μ. For every another point $y \in X_\mu$, $y \neq x$, there exists a pair of μ-open disjoint set U, V (see 6)such such that $x \in U$ and $y \in V$. It follows that $U \cap c_\mu(V) = \emptyset$. We conclude that x is μ_θ-closed and, consequently, X_{μ_θ} is μT_1-space. □

Proposition 5.1. *If X_μ is a μT_2 and $w\mu$-compact space, then each family of μ_θ-closed subsets with the finite intersection property has non-empty intersection.*

Proof. If $\{F_\alpha : \alpha \in \Lambda\}$ is a family of μ_θ-closed subsets with the finite intersection property, then $\{F_\alpha : \alpha \in \Lambda\}$ is a family of μ-closed subsets of X_{μ_θ} with the finite intersection property. Thus, $\cap\{F_\alpha : \alpha \in \Lambda\}$ is non-empty since X_{μ_θ} is $\mu-$quasi-compact, [9]. □

Theorem 8. *X_μ is a μ T_2 and $w\mu$-compact space if and only if X_{μ_θ} is μT_1 and μ-compact space.*

Proof. **If**. Let X_{μ_θ} be μT_1 and μ-compact space and let $\{F_a : a \in A\}$be a family of μ_θ-closed subsets of X_μ with the finite intersection property. Now $\{F_a : a \in A\}$ is the family of μ-closed subsets of X_{μ_θ} which is μT_1 and μ-compact space. It follows that the family $\{F_a : a \in A\}$has non-empty intersection. Thus, X_μ is a $w\mu$-compact space.

Only if. If $\{F_a : a \in A\}$ is a family of μ-closed subsets of $\{F_a : a \in A\}$ with the finite intersection property, then $\{F_a : a \in A\}$ is a family of μ_θ-closed subsets of X_μ with the finite intersection property. Since X_μ is $w\mu$-compact space, we infer that $\cap\{F_a : a \in A\}$ is non-empty. By this we infer that X_{μ_θ} μ-compact space. □

The following theorems are the main results of this section.

Theorem 9. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty $w\mu$-compact μT_2 spaces and $\mu-$closed mappings p_{ab}. If $X = \lim X$, then:*

a) For all $a \in A$

$$p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\},$$

b) If $X_a \neq \emptyset$ *for every* $a \in A$*, then* $X \neq \emptyset$*.*

Proof. Now inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is inverse system

$$\mathbf{X}_\mu = \{X_{a_{\mu_\theta}}, p_{ab}, A\}$$

of $\mu-$ quasi-compact spaces and closed mappings p_{ab}. Using Theorem 2 we complete the proof. □

Lemma 5. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of* $w\mu$*-compact* μT_2 *non-empty spaces and* $\mu-$*closed surjective bonding mapping* p_{ab}*. Then the projections* $p_a : \lim \mathbf{X} \to \mathbf{X}_a, a \in A,$ *are surjective and* $\mu-$*closed.*

Proof. The inverse system $\mathbf{X}_\mu = \{X_{a_{\mu_\theta}}, p_{ab}, A\}$ satisfies the condition of Theorem 2 by which we complete the proof. □

Theorem 10. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of* $w\mu$*-compact* μT_2 *non-empty spaces and* μ*-closed surjective bonding mapping* p_{ab}*. Then* $\lim \mathbf{X}$ *is* $w\mu$*-compact.*

Proof. By Theorem 2 the inverse system $\mathbf{X}_\mu = \{X_{a_{\mu_\theta}}, p_{ab}, A\}$ of $\mu-$quasi-compact spaces has $\mu-$quasi-compact limit $\lim \mathbf{X}_\mu$. This means that $\lim \mathbf{X}$ *$w\mu$-compact space.* □

Definition 22. We say that the mapping $f : X \to Y$ has the inverse property if $f^{-1}(\mathrm{c}_\mu V) = \mathrm{c}_\mu f^{-1}(V)$ for every μ-open set $V \subset Y$.

Now we shall prove the following result.

Theorem 11. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of* $w\mu$*-compact spaces such that* $\lim \mathbf{X}$ *is* $w\mu$*-compact and the projections* $p_a : \lim \mathbf{X} \to \mathbf{X}_a$*,* $a \in A$*, are surjective with the inverse property. If the spaces* X_a *are* μ*-connected, then* $X = \lim \mathbf{X}$ *is* μ*-connected.*

Proof. Suppose that X is not $\mu-$connected. It follows that there exist μ-closed sets F_1, F_2 such that

$$F_1 \cup F_2 = X \tag{1}$$

and

$$F_1 \cap F_2 = \emptyset. \tag{2}$$

For each $a \in A$ let $U_a \subset X_a$ be a maximal μ-open set such that $p_a^{-1}(U_a) \subset F_1$.Similarly, let $V_a \subset X_a$ be a maximal μ-open set such that $p_a^{-1}(V_a) \subset F_2$. From the *$w\mu$-compactness of* X it follows that there exist finitely many $U_{a1}, ..., U_{an}, V_{b1}, ..., V_{bm}$

$$X = \cup_{i=1}^{n} c_\mu(p_{ai}^{-1}(U_{ai})) \cup (\cup_{j=1}^{m} c_\mu(p_{bj}^{-1}(U_{bj}))).$$

We may use that $a_i = b_j$. This means that

$$X = \cup_{i=1}^{n} c_\mu(p_{ai}^{-1}(U_{ai})) \cup (\cup_{j=1}^{m} c_\mu(p_{aj}^{-1}(U_{aj}))).$$

It follows from the inverse property of the projections it follows

$$X = p_a^{-1}(\cup_{i=1}^{n} c_\mu(U_{ai}) \cup \ \cup_{j=1}^{m} (c_\mu(U_{aj})).$$

Now

$$X_a = \cup_{i=1}^{n} c_\mu(U_{ai}) \cup \ \cup_{j=1}^{m} (c_\mu(U_{aj}).$$

From $F_1 \cap F_2 = \emptyset$ it follows that

$$\cup_{i=1}^{n} c_\mu(U_{ai}) \cap \ \cup_{j=1}^{m} (c_\mu(U_{aj}) = \emptyset.$$

This is impossible since X_a is μ-connected. Thus $X = \lim \mathbf{X}$ is μ-connected. □

Since each open mapping has the inverse property we have the following result.

Theorem 12. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of $w\mu$-compact spaces such that* $\lim \mathbf{X}$ *is $w\mu$-compact and the projections* $p_a : \lim \mathbf{X} \to \mathbf{X}_a$, $a \in A$, *are surjective and open. If the spaces* X_a *are μ-connected, then* $X = \lim \mathbf{X}$ *is μ-connected.*

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