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FRACTIONAL GENERALIZATION OF TEMPERATURE FIELD PROBLEM IN OIL STRATA

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Abstract

We deal with temperature field problem for fractional lumped radial formulation in oil strata. To solve the boundary value problem for the fractional heat equation the method of integral transforms, namely, Laplace and Hankel transforms together with the modified form of convolution theorem known as Efro's theorem is used. Use of Caputo's differintegration operator provides new integral form of the solution.

1. Introduction

An oil stratum is taken to be a porous medium (sand stone) which is saturated with oil. The depth of an oil strata varies from one to several kilometers thus we can assume that the stratum's depth is equal to infinity. The rock surrounding a stratum (cap and base rock) is considered impermeable to the fluid.

Model for oil extraction: A standard method of oil extraction is to drill injection wells along the boundary of the oil reservoir and to inject hot fluid (water or steam) into these wells. The oil is pumped out from a series of production wells which are drilled in the centre of the oil deposit.

The problem arises, of describing the temperature field $u = u(x, y, z, t)$ in a single or multiple-layer oil stratum when the hot fluid whose temperature differs from that of the stratum, is injected into the stratum during the oil extraction process. The heat equation for a porous medium is derived under the following general assumptions on the model (Antimirov et al 1993).

Assumptions

(i) The temperature of the porous medium is equal to the temperature of the fluid which fills the strata,

(ii) The thermal properties of both the cap and the base rock are identical,

(iii) The boundary temperature and heat fluxes are equal on each side of the interface between the strata and its surrounding media,

(iv) The strata is horizontal and has constant depth,

(v) The consideration of the temperature field is restricted to the cap rock only and by choosing $z = 0$ the temperature on the surface of the strata is obtained.

Two cases of fluid injection, linear and radial, are mainly considered. Besides the exact formulation of the problem, three approximate formulations are treated (Antimirov et.al 1993). The lumped formulation where the stratum's thermal conductivity is infinitely large in vertical direction and is finite in the horizontal direction. The cap and base rock is considered to be thermally isotropic.The incomplete lumped formulation, where the horizontal heat transfer in the cap and the base rock is neglected, and the formulation of Lauwerier where horizontal heat transfer in the strata is also neglected.

The lumped formulation in the radial case, for the temperature field in a single oil stratum in dimensionless variable has the form [1]:

$$\frac{1}{a^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}, \quad 0 < r, z, t < \infty, \tag{1.1}$$

subject to the boundary condition

$$z = 0: \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1-2\nu}{r}\frac{\partial u}{\partial r} + \alpha\frac{\partial u}{\partial z}, \quad 0 < r, t < \infty \tag{1.2}$$

and the conditions

$$\begin{array}{ll} \text{(a)} & r = 0,\ z = 0 : u = 1 \\ \text{(b)} & \text{if } r^2 + z^2 \to \infty \text{ then } u = 0 \\ \text{(c)} & t = 0 : u = 0. \end{array} \tag{1.3}$$

- The constant $a > 0$ depends on the coefficient of thermal diffusivity of the cap rock and the strata
- The constant $\alpha > 0$ is a ratio of the coefficients of thermal conductivity of the cap rock and the strata.
- The constant $\nu > 0$ depends on the volume rate and the volumetric heat capacity of the fluid as well as the coefficient of the thermal conductivity of the strata.

Ben Nakhi and Kalla [2] have studied some boundary value problems of temperature field in oil strata by using the Laplace and the general Hankel transform. Recently Boyadjiev and Scherer [3] have considered some fractional extensions of temperature field problems in oil strata in the case of linear lumped and incomplete lumped formulation and radial incomplete lumped formulation.

In this paper we consider the fractional generalization of problem (1.1) to (1.3) giving radial case of the fractional lumped formulation.

The temperature field $u = u(r, z, t)$ satisfies the following fractional diffusion equation (Mainardi 1966) in the radial case

$$D_*^{2\beta} u = a^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\Big), \quad 0 < r, z, t < \infty, \ \ 0 < \beta \le 1/2 \quad (1.4)$$

subject to the boundary condition

$$z = 0\colon D_*^{2\beta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1-2\nu}{r}\frac{\partial u}{\partial r} + \alpha\frac{\partial u}{\partial z}, \quad 0 < r,\ t < \infty \quad (1.5)$$

and the conditions

(a) $r = z = 0 : u = 1$

(b) If $r^2 + z^2 \to \infty$ then $u \to 0$ and (1.6)

(c) $t = 0 : u = 0$

where we shall use the definition of fractional derivatives by Caputo as given in Podlubny [8]

$$D_*^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(x)\, dx}{(t-x)^{\alpha-m+1}}, & m-1 < \alpha < m,\ m \in \mathbf{N} \\ \dfrac{d^m f(t)}{dt^m}, & \alpha = m. \end{cases} \quad (1.7)$$

2. Useful Results

The following rule of the Laplace transform of fractional derivative will play an important role [8]

$$L\big[D_*^{\alpha} f(t)\big] = p^{\alpha} L\big(f(t)\big) - \sum_{k=0}^{m-1} f^k(0) p^{\alpha-1-k}, \quad m-1 < \alpha \le m. \quad (2.1)$$

The Laplace transform of the temperature function $u(r, z, t)$ is defined as [9]

$$\overline{u}(r, z, p) = L\big\{u(r, z, t); t \to p\big\} = \int_0^{\infty} e^{-pt} u(r, z, t)\, dt, \quad \mathrm{Re}\,(p) > 0 \quad (2.2)$$

and inversion formula for Laplace transform gives

$$L^{-1}\left[\overline{u}(r,z,p)\right] = u(r,z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,\overline{u}(r,z,p)\,dp\,, \tag{2.3}$$

Efro's theorem [1]: Let be given analytic functions $G(p)$ and $q(p)$ and the relations

$$F(p) = L\{f(t)\} \tag{2.4}$$

and

$$\overline{e}^{\tau q(p)}G(p) = L\left[g(t,\tau)\right] \tag{2.5}$$

then

$$G(p)\,F\big(q(p)\big) = L\left[\int_0^\infty f(\tau)\,g(t,\tau)\,d\tau\right]. \tag{2.6}$$

This theorem is a simple generalization of the convolution theorem for the Laplace transform, if we put $q(p) = p$.

Following auxiliary function of Wright's type helps in expressing the solution for the time fractional diffusion equation [7]

$$M(z;\beta) = \frac{1}{2\pi i}\int_{H_a} e^{\sigma - z\sigma^\beta}\,\frac{d\sigma}{\sigma^{1-\beta}}\,, \quad 0 < \beta < 1\,, \tag{2.7}$$

where H_a denotes the Hankel path of integration that begins at $\sigma = -\infty - ib_1$ $(b_1 > 0)$, encircles the branch point that lies along the negative real axis, and ends up at $-\infty + ib_2$ $(b_2 > 0)$. It is also proved that following relation takes place

$$M(z;\beta) = W(-z;-\beta;1-\beta)\,, \tag{2.8}$$

where

$$\begin{aligned} W(z;\lambda,\mu) &= \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)} \\ &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma + z\sigma^{-\lambda}}\,\frac{d\sigma}{\sigma\mu}\,, \quad \lambda > -1,\ \mu > 0\,, \end{aligned} \tag{2.9}$$

is an entire function of z referred to as the Wright's function [4, Vol.III, Ch.18]. In the particular case $\beta = \frac{1}{2}$ it gives

$$M\Big(z;\frac{1}{2}\Big) = \frac{1}{\sqrt{\pi}}\sum_{m=0}^{\infty}(-1)^m\Big(\frac{1}{2^2}\Big)^m\frac{z^{2m}}{m!} = \frac{1}{\sqrt{\pi}}\,e^{-z^2/4}\,, \tag{2.10}$$

3. Fractional Lumped Formulation: The radial case

Theorem

The solution of the problem posed in equations (1.4), (1.5) and (1.6) in the radial case of the fractional lumped formulation is given by

$$u(r,z,t)=\frac{1}{\Gamma(2\nu)}\int_0^\infty \frac{\tau^{2\nu}}{R}\,e^{-\alpha\tau}\,\psi(R,t)\,dt \tag{3.1}$$

where

$$\begin{aligned}\psi(R,t)=&\alpha^{2\nu}+\frac{1}{\pi}\int_0^\infty \frac{e^{-\rho t-R\rho^\beta\cos\pi\beta}}{\rho}(\alpha^2+p^{2\beta}+2\alpha\rho^\beta\cos\pi\beta)^\nu\\ &\times\sin(2\nu\phi-R\rho^\beta\sin 2\pi\beta)d\rho,\quad \text{when } 2\nu \text{ is not an integer}\end{aligned} \tag{3.2}$$

with

$$R=\sqrt{(z+\tau)^2+r^2};\ \phi=\tan^{-1}\left(\frac{\rho^\beta\sin\pi\beta}{\alpha+\rho^\beta\cos\pi\beta}\right).$$

When 2ν is an integer then

$$\psi(R,t)=e^{\alpha R}\frac{d^n}{dR^n}\left[e^{-\alpha R}\int_0^t \frac{R\beta}{u^{\beta+1}}M\left(\frac{R}{u^\beta};\beta\right)du\right],\quad \text{for } 0<\beta\le 1/2\,. \tag{3.3}$$

Proof: Let us apply Laplace transform to (1.4), (1.5) and (1.6) and use the result (2.1) to give

$$p^{2\beta}\overline{u}=a^2\left(\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}+\frac{\partial^2\overline{u}}{\partial z^2}\right);\quad 0<r,\ z<\infty\,, \tag{3.4}$$

$$z=0\colon p^{2\beta}\,\overline{u}=\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1-2\nu}{r}\frac{2\overline{u}}{\partial r}+\alpha\frac{\partial\overline{u}}{\partial z},\quad 0<r<\infty\,, \tag{3.5}$$

(a) $r=z=0;\ \overline{u}=\frac{1}{p}$

(b) If $r^2+z^2\to\infty$ then $\overline{u}\to 0$, (3.6)

Applying the Hankel transform of order zero [10]

$$\tilde{\overline{u}}(\lambda,z,p)=\int_0^\infty \overline{u}(r,z,p)rJ_0(\lambda r)\,dr\,, \tag{3.7}$$

and the following property

$$\int_0^\infty \frac{\partial}{\partial r}\left(r\,\frac{\partial \overline{u}}{\partial r}\right) J_0(\lambda r)\,dr = -\lambda^2 \tilde{\overline{u}}(\lambda, z, p) \tag{3.8}$$

in the equation (3.4) it reduces to the form

$$\frac{d^2}{dz^2}\,\tilde{\overline{u}} - \left(\lambda^2 + \frac{p^{2\beta}}{a^2}\right)\tilde{\overline{u}} = 0\,. \tag{3.9}$$

The solution to the above equation which remains bounded as $z \to \infty$ has the form:

$$\tilde{\overline{u}}(\lambda, z, p) = C(\lambda, p)\, e^{-z\sqrt{\lambda^2 + p^{2\beta}/a^2}}\,. \tag{3.10}$$

Now $C(\lambda, p)$ is to be determined by using inverse Hankel transform to (3.10), we have

$$\tilde{\overline{u}}(r, z, p) = \int_0^\infty \lambda C(\lambda, p)\, e^{-z\sqrt{\lambda^2 + p^{2\beta}/a^2}}\, J_0(\lambda r)\, d\lambda\,. \tag{3.11}$$

Putting (3.11) in boundary condition (3.5)

$$\int_0^\infty p^{2\beta}\lambda C(\lambda, p) J_0(\lambda r)\, d\lambda = \int_0^\infty \left(\frac{\partial^2}{\partial r^2} + \frac{1-2\nu}{r}\,\frac{\partial}{dr} + \alpha\,\frac{\partial}{\partial z}\right)\lambda C(\lambda, p) J_0(\lambda r)\, d\lambda\,. \tag{3.12}$$

Now using following results in (3.12)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\right) J_0(\lambda r) = -\lambda^2 J_0(\lambda r)$$

and

$$\frac{\partial}{\partial r}\, J_0(\lambda r) = -\lambda J_1(\lambda r)\,, \tag{3.13}$$

it reduces to the form

$$\begin{aligned} &\int_0^\infty r C_1(\lambda, p)\left(\lambda^2 + p^{2\beta} + \alpha\sqrt{\lambda^2 + \frac{p^{2\beta}}{a^2}}\right) J_0(\lambda r)\, d\lambda \\ &= 2\nu \int_0^\infty \lambda C_1(\lambda, p) J_1(\lambda r)\, d\lambda\,, \end{aligned} \tag{3.14}$$

where $C_1(\lambda, p) = \lambda C(\lambda, p)$. Substituting

$$\frac{1}{\lambda}\, C_1(\lambda, p)\left(\lambda^2 + p^{2\beta} + \alpha\sqrt{\lambda^2 + \frac{p^{2\beta}}{a^2}}\right) = f(\lambda, p)\,, \tag{3.15}$$

(3.14) reduces to the form

$$\int_0^\infty r\lambda f(\lambda,p)J_0(\lambda r)\,d\lambda = 2\nu\int_0^\infty \lambda C_1(\lambda,p)J_1(\lambda r)\,d\lambda\,, \tag{3.16}$$

multiplying both the sides of the last equation by $J_0(xr)$ then integrating with respect to r from 0 to ∞ and interchanging the order of integration in the right hand term

$$\int_0^\infty rJ_0(rx)\,dr\int_0^\infty \lambda f(\lambda,p)J_0(\lambda r)\,d\lambda = 2\nu\int_0^\infty \lambda C_1(\lambda,p)\,d\lambda\int_0^\infty J_0(rx)J_1(\lambda r)\,dr\,. \tag{3.17}$$

Now using the well known Fourier Bessel integral formula

$$\int_0^\infty rJ_0(rx)\,dr\int_0^\infty \lambda f(\lambda,p)J_0(\lambda r)\,d\lambda = f(x,p) \tag{3.18}$$

and a special case of the discontinuous Weber Schafheintlin integration formula [11, p.411]

$$\int_0^\infty J_0(rx)J_1(\lambda x)\,dr = \begin{cases} \dfrac{1}{\lambda}\,, & \lambda > x \\ 0\,, & \lambda < x \end{cases} \tag{3.19}$$

into the equation (3.17), we get

$$f(x,p) = 2\nu\int_0^\infty C_1(\lambda,p)\,d\lambda\,. \tag{3.20}$$

From (3.15) and (3.20), we get

$$\frac{C_1(x,p)}{x}\left(x^2+p^{2\beta}+\alpha\sqrt{x^2+\frac{p^{2\beta}}{a^2}}\right) = 2\nu\int_x^\infty C_1(\lambda,p)\,d\lambda\,. \tag{3.21}$$

Let us assume

$$\phi(x,p) = \int_x^\infty C_1(\lambda,p)\,d\lambda\,, \tag{3.22}$$

then

$$\frac{d\phi}{dx} = -C_1(x,p)\,,\quad \text{since}\quad \underset{x\to\infty}{Lt}\; C_1(x,p) = 0\,. \tag{3.23}$$

Now replacing (3.22), (3.23) into (3.21) we shall get following ordinary differential equation when $a^2 = 1$

$$\frac{d\phi}{dx} + \frac{2\nu x}{\left(x^2+p^{2\beta}+\alpha\sqrt{x^2+p^{2\beta}}\right)}\,\phi = 0\,. \tag{3.24}$$

Solution of (3.24) under the given initial condition (3.6a) is

$$\phi(x,p)=\frac{(\alpha+p^{\beta})^{2\nu}}{p\left(\alpha+\sqrt{x^2+p^{2\beta}}\right)^{2\nu}}\,. \tag{3.25}$$

Now solving (3.23), (3.24) with the help of (3.25), we have

$$C_1(x,p)=\frac{2\nu x(\alpha+p^{\beta})^{2\nu}}{p\sqrt{(x^2+p^{2\beta})}\left(\alpha+\sqrt{x^2+p^{2\beta}}\right)^{2\nu+1}}\,. \tag{3.26}$$

Now using relation $C_1=\lambda C$, and (3.26) into the equation (3.11), we get

$$\overline{u}(r,z,p)=\int_0^\infty \frac{2\nu\lambda(\alpha+p^{\beta})^{2\nu}\,e^{-z\sqrt{\lambda^2+p^{2\beta}}}}{p\sqrt{\lambda^2+p^{2\beta}}\left(\alpha+\sqrt{\lambda^2+p^{2\beta}}\right)^{2\nu+1}}\,J_0(\lambda r)\,d\lambda\,. \tag{3.27}$$

To find inverse Laplace transform of $\overline{u}$, we write

$$\overline{\phi}(\lambda,z,p;\beta)=\frac{2\nu\lambda(\alpha+p^{\beta})^{2\nu}\,e^{-z\sqrt{\lambda^2+p^{2\beta}}}}{p\sqrt{\lambda^2+p^{2\beta}}\left(\alpha+\sqrt{\lambda^2+p^{2\beta}}\right)^{2\nu+1}}\,, \tag{3.28}$$

and apply Efro's theorem. First we express

$$\overline{\phi}(\lambda,z,p;\beta)=G(p;\beta)\,F\big[q(p;\beta)\big]\,, \tag{3.29}$$

where

$$G(p;\beta)=\frac{2\nu\lambda(\alpha+p^{\beta})^{2\nu}}{p\sqrt{\lambda^2+p^{2\beta}}}\,e^{-z\sqrt{(\lambda^2+p^{2\beta})}}\,, \tag{3.30}$$

$$q(p;\beta)=\alpha+\sqrt{\lambda^2+p^{2\beta}}\,, \tag{3.31}$$

and

$$F(p)=\frac{1}{p^{2\nu+1}}=L\left[\frac{t^{2\nu}}{\Gamma(2\nu+1)}\right]. \tag{3.32}$$

Now

$$\begin{aligned}G(p;\beta)\,e^{-\tau q(p;\beta)}&=\frac{2\nu\lambda(\alpha+p^{\beta})^{2\nu}}{p\sqrt{\lambda^2+p^{2\beta}}}\,e^{-z\sqrt{\lambda^2+p^{2\beta}}}\,e^{-\tau\left(\alpha+\sqrt{\lambda^2+p^{2\beta}}\right)}\\&=\frac{2\nu\lambda(\alpha+p^{\beta})^{2\nu}}{p\sqrt{\lambda^2+p^{2\beta}}}\,e^{-\tau\alpha}\,e^{-(z+\tau)\sqrt{\lambda^2+p^{2\beta}}}\,,\\&=L\big[g(\lambda,t,\tau;\beta)\big]\,.\end{aligned} \tag{3.33}$$

Using inverse formula for Laplace transform

$$g(\lambda,t,\tau,\beta) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{2\lambda\nu(\alpha+p^{\beta})^{2\nu}}{p\sqrt{\lambda^2+p^{2\beta}}}\, e^{-\tau\alpha}\, e^{-(z+\tau)\sqrt{\lambda^2+p^{2\beta}}}\, e^{pt}\, dp\,. \quad (3.34)$$

By Efro's theorem

$$L^{-1}\left[\overline{\phi}(\lambda,z,p;\beta)\right] = \int_0^{\infty} f(\tau)\, g(\lambda,t,\tau;\beta)\, d\tau \quad (3.35)$$

$$= \int_0^{\infty} \frac{\tau^{2\nu}}{\Gamma(2\nu+1)}\, d\tau\, \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{2\nu\lambda(\alpha+p^{\beta})^{2\nu}}{p\sqrt{\lambda^2+p^{2\beta}}}\, e^{-\tau\alpha}\, e^{-(z+\tau)\sqrt{\lambda^2+p^{2\beta}}}\, e^{pt}\, dp\,.$$

Now, taking inverse Laplace transform of (3.27), interchanging the order of integration in the right hand side and using (3.35) we get

$$\begin{aligned} u(r,z,t) = {} & \frac{1}{2\pi i}\int_0^{\infty} \tau^{2\nu}\, \frac{e^{-\alpha\tau}}{\Gamma(2\nu)}\, d\tau \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\alpha+p^{\beta})^{2\nu}}{p}\, e^{pt}\, dp \\ & \times \int_0^{\infty} \frac{\lambda\, e^{-(z+\tau)\sqrt{\lambda^2+p^{2\beta}}}}{\sqrt{\lambda^2+p^{2\beta}}}\, J_0(\lambda r)\, d\lambda\,. \end{aligned} \quad (3.36)$$

Using following formula [5, p.9, formula 24]

$$\int_0^{\infty} \frac{x\, e^{-\alpha\sqrt{x^2+p^{2\beta}}}}{\sqrt{x^2+p^{2\beta}}}\, J_0(xr)\, dx = \frac{e^{-p^{\beta}\sqrt{\alpha^2+r^2}}}{\sqrt{\alpha^2+r^2}} \quad (3.37)$$

(3.36) becomes

$$u(r,z,t) = \frac{1}{\Gamma(2\nu)}\int_0^{\infty} \frac{\tau^{2\nu}}{R}\, e^{-\alpha\tau}\, \psi(R,t)\, d\tau\,, \quad (3.38)$$

where

$$R = \sqrt{(z+\tau)^2+r^2}\,, \quad (3.39)$$

$$\psi(R,t) = L^{-1}\left\{\frac{(\alpha+p^{\beta})^{2\nu}\, e^{-p^{\beta}R}}{p}\right\} \quad (3.40)$$

$$= L^{-1}\left[\overline{\psi}(R,p)\right] \quad \text{(say)} \quad (3.41)$$

The inverse Laplace transform of $\overline{\psi}$ can be found by deforming the contour of integration in the inversion formula as shown in figure 1.

Case I: If we take 2ν not an integer:

The origin $p = 0$ is a branch point of the function $\overline{\psi}(R,p)$. Hence each branch of the function $\overline{\psi}(R,p)$ is single valued analytic function inside the closed contour shown in the fig.1.

Since 2ν is not an integer we use the formula

$$(\alpha + p^{\beta})^{2\nu} = e^{2\nu \log(\alpha + p^{\beta})}\,; \quad 0 < \beta \le \frac{1}{2}\,.$$

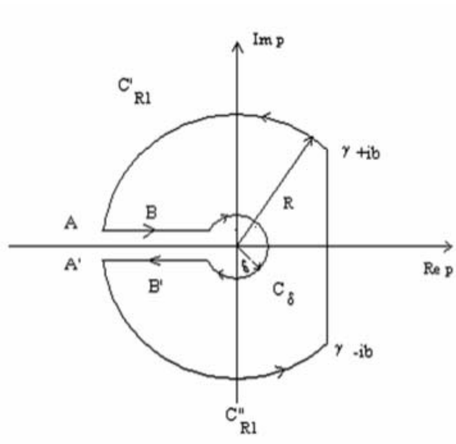

Figure 1

Integration contour

The singular point $p = \alpha^{1/\beta}\, e^{\pm\pi i/\beta}$, is not situated inside the closed contour because in this region $|\arg p| \le \pi$. Hence by Cauchy residue theorem we have

$$\frac{1}{2\pi i}\left[\int_{\gamma-ib}^{\gamma+ib} + \int_{C'_{R_1}} + \int_{AB} + \int_{C_\delta} + \int_{B'A'} + \int_{C''_{R_1}}\right]\overline{\psi}(R,p)\, e^{pt}\, dp = 0\,, \qquad (3.42)$$

It is easily seen that

$$\lim_{R_1\to\infty}\overline{\psi}(R,p)\Big|_{p\in C'_{R_1}} = 0\,, \quad \lim_{R_1\to\infty}\overline{\psi}(R,p)\Big|_{p\in C''_{R_1}} = 0$$

hence

$$\lim_{R_1\to\infty}\int_{C'_{R_1}}\overline{\psi}(R,p)\, e^{pt}\, dp = 0\,, \quad \lim_{R_1\to\infty}\int_{C''_{R_1}}\overline{\psi}(R,p)\, e^{pt}\, dp = 0\,. \qquad (3.43)$$

Now to compute the integral along segments AB, $B'A'$ and small circle C_δ.

(i) For segment AB we have $p = \rho\, e^{i\pi}$, $p^\beta = \rho^\beta\, e^{i\beta\pi}$, $0 < \beta \leq \frac{1}{2}$, $dp = -dp$ then

$$\int_{AB} \overline{\psi}(R,p)\, e^{pt}\, dp = \int_\infty^0 \frac{(\alpha + \rho^\beta\, e^{i\beta\pi})^{2\nu}}{\rho} \exp\{-R\rho^\beta\, e^{i\beta\pi}\}\, e^{-\rho t}\, d\rho\,. \tag{3.44}$$

(ii) Similarly on $B'A'$, $p = \rho\, e^{i\pi}$ and

$$\int_{B'A'} \overline{\psi}(R,p)\, e^{pt}\, dp = \int_0^\infty \frac{(\alpha + \rho^\beta\, e^{-i\beta\pi})^{2\nu}}{\rho} \exp(-R\rho^\beta\, e^{i\beta\pi})\, e^{\rho\tau}\, d\rho. \tag{3.45}$$

Then sum of (i) and (ii) after some simplification is represented as follows

$$\frac{1}{2\pi i}\left[\int_{AB} + \int_{B'A'}\right] \overline{\psi}(r,p)\, e^{pt}\, dp =$$

$$-\frac{1}{\pi}\int_0^\infty \frac{e^{\rho t}}{\rho} e^{-R\rho^\beta \cos\pi\beta} (\alpha^2 + \rho^{2\beta} + 2\alpha\rho^\beta \cos\pi\beta)^\nu \sin(2\nu\phi - R\rho^\beta \sin\pi\beta)\, d\rho$$

where

$$R = \sqrt{(z+\tau)^2 + r^2};\ \phi = \tan^{-1}\left(\frac{\rho^\beta \sin\pi\beta}{\alpha + \rho^\beta \cos\pi\beta}\right). \tag{3.46}$$

(iii) On circle C_δ, $p = \delta\, e^{i\phi}$, $dp = i\delta\, e^{i\phi}\, d\phi$

$$\frac{1}{2\pi i}\int_{C_\delta} \frac{(\alpha + p^\beta)^{2\nu}}{p} e^{-Rp^\beta} e^{pt}\, dp =$$

$$= \frac{1}{2\pi i}\int_\pi^{-\pi} (\alpha + \delta^\beta\, e^{i\beta\phi})^{2\nu}\, e^{-R\delta^\beta\, e^{i\beta\phi}}\, i\, d\phi = -\alpha^{2\nu}\,, \quad \text{as} \quad \delta \to 0\,. \tag{3.47}$$

Use of the integrals from (3.43), (3.46) and (3.47) in (3.42) provides the following result

$$\psi(R,t) = \alpha^{2\nu} + \frac{1}{\pi}\int_0^\infty \frac{e^{-\rho t}}{\rho} e^{-R\rho^\beta \cos\pi\beta} (\alpha^2 + \rho^{2\beta} + 2\alpha\rho^\beta \cos\pi\beta)^\nu$$
$$\times \sin(2\nu\phi - R\rho^\beta \sin\pi\beta)\, d\rho\,, \tag{3.48}$$

where R and ϕ are given by equation (3.46).

Case II. When $2\nu = n$ is an integer then (3.38) is represented as

$$u(r,z,t) = \int_0^\infty \frac{\tau^n}{\Gamma(n)} \frac{e^{-\alpha\tau}}{R}\, d\tau \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\alpha+p^\beta)^n}{p} e^{-p^\beta R} e^{pt}\, dp$$

$$= \int_0^\infty \frac{\tau^n}{\Gamma(n)} \frac{e^{-\alpha(\tau-R)}}{R}\, d\tau \Big[\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\alpha+p^\beta)^n}{p} e^{-(\alpha+p^\beta)R} e^{pt}\, dp\Big]$$

$$= \int_0^\infty \frac{\tau^n}{\Gamma(n)} \frac{e^{-\alpha(\tau-R)}}{R} \frac{d^n}{dR^n} \Big[\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-(\alpha+p^\beta)R}}{p} e^{pt}\, dp\Big]$$

$$= \int_0^\infty \frac{\tau^n}{\Gamma(n)} \frac{e^{-\alpha(\tau-R)}}{R} \frac{d^n}{dR^n} \Big[L^{-1}\Big(\frac{e^{-p^\beta R}}{p}\Big) e^{-\alpha R}\Big] d\tau \tag{3.49}$$

Now the $L^{-1}\{e^{-p^\beta R}/p\}$ is a direct consequence of the result given in [7] and a property of Laplace transform to give

$$L^{-1}\Big\{\frac{e^{-p^\beta R}}{p}\Big\} = \int_0^t \frac{R\beta}{u^{\beta+1}} M\Big(\frac{R}{u^\beta}\,;\beta\Big)\, du \tag{3.50}$$

where $M(z;\beta)$ is an auxiliary function of Wright's type defined by (2.7).

Special Case

If we take $\beta = 1/2$ in our problem (1.4), (1.5) and (1.6) we arrive at the problem (1.1), (1.2), (1.3) discussed by Antimirov [1]. The solution as given in (1, p.184, eq.8.3.49) can directly be obtained by substitution $\beta = 1/2$ in the result (3.48), when 2ν is not an integer. In the case when $2\nu = n$ is an integer the result (3.50) reduces in the form, using (2.10)

$$L^{-1}\Big[\frac{e^{-R\sqrt{p}}}{p}\Big] = \int_0^t \frac{R}{2u^{3/2}} M\Big(\frac{R}{\sqrt{u}};\, \frac{1}{2}\Big)\, du = \int_0^t \frac{R}{2\sqrt{\pi}\, u^{3/2}} e^{-R^2/4u}\, du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^t e^{-(R/2\sqrt{u})^2}\, d\Big(\frac{R}{2\sqrt{u}}\Big) = \frac{2}{\sqrt{\pi}} \int_{R/2\sqrt{t}}^\infty e^{-\omega^2}\, d\omega = \operatorname{erfc}\Big(\frac{R}{2\sqrt{t}}\Big)$$

and (3.49) gives the known result [1, p.182, eq. (8.3.41)].

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ДЕЛУМНА ГЕНЕРАЛИЗАЦИЈА НА ПРОБЛЕМОТ НА ТЕМПЕРАТУРНО ПОЛЕ ВО "OIL STRATA"

Mridula Garg *, Alka Rao *, S. L. Kalla **

Р е з и м е

Работиме на проблемот на температурно поле за делумно збиена радијална формулација на "Oil Strata". За да го решиме проблемот на гранична вредност за парцијална топлоттна равенка, се користи методот на интегрални трансформации, Laplace-ови Hankel-ови трансформации заедно со модифицираната форма на теоремата позната како Efro-ва теорема.

Употребата на Caputo-овиот диферо-интеграционаен оператор обезбедува нова интегрална форма на решението.

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