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# GENERALIZATIONS OF B*-ALGERAS

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This is a brief survey of the work done during the last two decades on generalizations of B*-algebras. We concentrate on only two aspects of the theory: the structure theorems and the numerical ranges. The Survey includes some recent (not yet published) results obtained by the author's student Mr. S. J. Bhatt, whose help the author gratefully acknowledges in preparing this article.

By a (*locally convex*) *topological algebra* A (assumed to be with identity 1), we mean a (locally convex) topological vector space which is also an algebra with separately continuous multiplication. A *locally m-convex* (*lmc*) *algebra* A (due to Michael [19]) is a locally convex algebra in which $o$ has a neighbourhood subbase consisting of idempotents; or equivalently, there is a family $P = (p_\alpha)$ of seminorms determining the topology of $A$ and satisfying $p_\alpha(xy) \leqslant p_\alpha(x)\, p_\alpha(y)$. Recall that a complete lmc algebra $A$ is a projective limit of the Banach algebras.

## 1. STRUCTURE THEOREMS

The Gelfand-Naimark Theorem [4, Theorem 1.1.1] for commutative B*-algebra $(A, \|\cdot\|)$ represents it isometrically and* isomorphically onto $C(x)$, the supnorm Banach algebra of all continuous complex valued functions on a compact Hausdorff space $X$, the weakly topologized space of

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all characters of (automatically continuous) on A. An earliest generalization of this result to the setting of lmc algebras is due to Michael [19]. A *$b^*$-algebra,* which is an lmc analogue of $B^*$-algebras, is a complete lmc* algebra $A$ with a defining family $P=(p_\alpha)$ of submultiplicative seminorms $p_\alpha$ satisfying $p_\alpha(x^*x)=p_\alpha(x)^2\ (x \in A)$.

**Theorem 1.1.** Let $(A, r)$ be a commutative $b^*$-algebra, $\Sigma_A$ be the usual weakly topologized space of all non-zero continuons characters on $A$ and $(C(\Sigma_A), r_0(A))$ be the usual algebra of all continuous coplex valued functions on $\Sigma_A$ with the topology $r_0(A)$ of uniform convergence on compact equicontinuous subsets of $\Sigma_A$. Then

(*a*) $\Sigma_A$ is a completely regular Hausdorff space

and (*b*) $A$ is algebraically and topologically* isomorphic to

$$(C(\Sigma_A), r_0(A)).$$

This shows that the duality problems conecrning the structure of a commutative $b^*$-algebra $A$ and the topologcal structures of $\Sigma_A$ and $C(\Sigma_A)$ are fairly intricate. Quite a good deal of work has been done in this direction. We only mention an interesting paper of Apostol [3] which, among other things, contains the following:

**Theorem 1.2.** Let $A$ be a $b^*$-algebra with a defining $b^*$-calibration $P=(p_\alpha)$. Let $A_s=\{x \in A \mid p(x)=\sup_\alpha p_\alpha(x)<\infty\}$. Then $A_s$ is *a* *subalgebra of $A$, $(A_s, p)$ is *a* $B^*$-algebra which is dense in $A$. Further, if A is commutative and if $X_s$ is the usual weakly topologized space of all non-zero characters on $A$; then $X_A=\theta\Sigma_A$, the real compactification of $\Sigma_A$ and $X_{A_s}=\beta\Sigma_A$, the Stone-Cech compactification of $\Sigma_A$.

This shows that we have also the algebraic *isomorphism $A \cong C(X_A)$. In general, $\Sigma_A \neq X_A$; but if $A$ is an $F$-algebra, then we get following.

**Theorem 1.3.** Let $A$ be a commutative $b^*$-algebra which is also an $F$-algebra. Then $\Sigma_A=X_A$ and $A$ is algebrically and topologically *isomorphic to $(C(X_A), k)$ where **k** is the compact-open topology.

Another beautiful result of Apostol is the abstract characterization of the algebra of continuous functions on a locally compact space with the compact open topology. For an lmc *algebra $A$, let $P(A)$ be the collection of all continuous submultiplicative semi-norms on A. Let, for $p \in P(A)$,

$$I_p = \{x \in A : p\ (x) = 0\},\ \alpha I_p = \{x \in A : xI_p = \{0\}\}.$$

*A* $b^*$-algebra $A$ is called *perfect* if the ideal $\sum_{p \in p(A)} \alpha\ I_p$ is dense in $A$.

**Theorem 1.4.** In order for an lmc *algebra $A$ to be topologically i*isomorphic with $(C\ (X), k)$ for some locally compact Hausdorff space $X$, t is necessary and sufficient that $A$ be a perfect commutative b*-algebra (with unit).

Now we turn to the general non-commutative case. The well known *GNS* construction [4, § 1.6] associates a representation $\pi_f$ with a given positive functional f on a star algebra $A$; and if $A$ is Banach, then each $\pi_f(a)$ is a bounded operator. This yields a *homomorphism of $A$ into $\beta(H_f)$, the algebra of all bounded operators on some Hilbert space $H_f$; and additionally, if $A$ is a $B^*$-algebra, then these representations $\pi_f$, over varying $f$, can be summed upto get an isometric *isomorphism of $A$ into $\beta(H)$ for some Hilbert space H.

In [9], Brooks has studied the representations on an lmc* algebra, thereby generalizing a number of Banach algebra results. The following theorem summarizes some of his results:

**Theorem 1.5.** Let $A$ be a complete lmc *algebra with 1. Let $P_1\ (A)$ be the set of all continuous positive functionals on $A$ and let $K(A) = \{f \in P_1\ (A) \mid f\ (1) = 1\}$. Then $K(A)$ is a weak* closed convex subset of the dual $A'$; is the closed convex null of its extreme points; and an $f$ in $K\ (A)$ is an extreme point if and only if f is indecomposable.

It should be noted that unlike Banach *algebra, a positive linear functional on a complete lmc *algebra need not be continuous. The most general automatic continuity result known here is due to Dixon [11, Theorem 11.1].

**Teorem 1.6.** (*a*) Every positive linear functional on a complete metrizable topological *algebra with identity is continuous.

(*b*) Let $A$ be a sequentially complete bornological lmc *algebra with 1. Then every positive linear functional on $A$ is continuous.

However, the GNS construction with continuous positive functionals on a complete lmc *algebra [9] gives results analogous to the case of Banach *algebras.

**Theorem 1.7.** Let $A$ be a complete lmc *algebra.

(*a*) If $f$ is a continuous positive functional on $A$, then the GNS representation $\pi_f$ associated with $f$ defines a topologically cyclic representation of $A$ into $\beta(H_f)$.

Further, $\pi_f$ is irreducible if and only if $f$ is an extreme point of $K(A)$.

(*b*) A topologically cyclic representation $\pi$ of $A$ into bounded operators on some Hilbert space $H$ is continuous if and only if it is unitarily equivalent to $\pi_f$ for some $f$ in $K(A)$.

(*c*) A representation $\pi$ of $A$ into some $\beta(H)$ is continuous if and only if it is a direct sum of an uniformly bounded family of topologically cyclic representations.

Brooks has also obtained an integral representation of continuous positive functional on a commutative complete lmc* algebra $A$ with the help of which he has investigated the representations on $A$ in standard form.

The machinary developed in [9] has been applied in [10] to represent an $F^*$-algebra topologically and *isomorphically onto a certain *algebra of operators all defined on a common dense subspace of a Hilbert space $H$.

Brook's work made clear the lack of a concrete model against which a $b^*$-algebra can be compared. It also shows that the $b^*$-algebras are to some extent inadequate for the discussion of algebras of unbounded operators in a Hilbert space. A model more general than $b^*$-algebra has been suggested by Allan [2] and investigated by Dixon [12].

In [1] Allan developed a spectral theory for locally convex algebra based on his abstract notion of' bounded element' of $A$ and under a weaker completeness condition of 'pseudocomleteness'. In [2], he introduced 'Generalized $B^*$-algebras' ($GB^*$-algebras). A modified, more general definition due to Dixon [12] is as follows:

*A $GB^*$-algebra* is a topological *algebra with 1 such that

(*a*) $A$ admits a largest (under inclusion) bounded *semigroup $B_0$, called its unit ball, which is also closed and absolutely convex,

(*b*) the *subalgebra $A(B_0) = \{\alpha x \mid \alpha \in \mathbb{C},\ x \in B_0\}$ is a Banach algebra with the Minkowski functional $\|\cdot\|_{B_0}$ of $B_0$ as norm,

(*c*) $A$ is symmetric in the sense that for each $x$ in $A$, $(1 + x^*x)^{-1} \in A(B_0)$

In fact, the algebra $(A(B_0), \|\cdot\|_{B_0})$ turns out to be a $B^*$-algebra. It can be regarded to be the bounded part of $A$. Non trivial examples of $GB^*$-algebras include the Arens' algebra $L^w(X) = \bigcap_{1 \leqslant p < = \infty} L^p(X)$ on a finite measure space $X$, its σ-finite varient $L^w_{loc}(X)$, the algebra $\mathrm{m}(X)$ of measurable functions on $X$ and its non-commutative analogue viz. the algebra $\mathrm{m}(A)$ of operators in a Hilbert space which are locally measurable w.r.t. a von Neumann algebra $A$ [13].

Allan's paper [2] is centered around the following commutative Gelfand-Naimark type theorem (see also [12]).

**Theorem 1.8.** Let $A$ be a commutative $GB^*$-algebra with unit ball $B_0$. Then the Gelfand isomorphism $\varnothing$: $A(B_0) \to C(\Sigma_{A(B_0)})$ extends uniquely to a *isomorphism of $A$ onto a *algebra of functions on $\Sigma_{A(B_0)}$.

By a ***algebra of functions*** on a compact Hausforff space $X$ he means a set consisting of $\varnothing^* = \varnothing \cup \{\infty\}$ valued continuous functions on $X$ taking the value $\infty$ on atmost a nowhere dense subset of $X$ and forming a *algebra u der the natural operations. Another important result due to Allan is that every continuous hermition linear functional on a commutative $GB^*$-algebra $A$ is the difference of two (not necessarily continuous) positive functionals.

Dixon [12, § 5] developed a functional calculas for normal element in a $GB^*$-algebra $A$; and with the help of a largest locally convex $GB^*$-topology an A (whose existence he established in case $A$ is locally convex) analyzed the positive functionals on A. His analysis ultimately leads to the following Gelfand-Naimark type theorem.

**Theorem 1.9.** Let $A$ be a locally convex $GB^*$-algebra with unit ball $B_0$. Then there exists a faithful *representation of $A$ as an extended $C^*$-algebra $\mathcal{A}$ defined on a common domain $D$ dense in a Hilbert space H such that the elements in $A(B_0)$ bijectively corresponds to the operators in $\mathcal{A} \cap \beta(H)$.

By an extended $C^*$-algebra is ment the following:

Let H be a Hibert space. $A$ set $A$ of densely defined closed operators $T$ with domains $D(T)$ in $H$ is called a **algebra of closed operators* if it is a *algebra under the operations: $S+T=(s+T)^-, \alpha.T=(\alpha T)^-, S.T==(ST)^-$ and $T \to T^*$ (the adjoint of $T$). Here bar denoted the closure of the operator concerned. It is called an *extended $C^*$-algebra* if $A \cap \beta(H)$ is *a* $C^*$-algebra and $(I+T^*T)^{-1} \in A$ for all $T \in A$. Further, if $A \cap \beta(H)$ is a $W^*$-algebra, then $A$ is called an *extended $W^*$-algebra.* The algebra $A$ is said to have a common dense domain $D$ if $D = \bigcap \{D(T) \mid T \in A\}$ is dense in $H$.

## 2. NUMERICAL RANGES AND VIDAV-PALMER TYPE THEOREMS

Last decade witnessed an exciting development in the study of numerical ranges of Banach space operators and Banach algebra elements [7, 8]. *A* beautiful global result is the Vidav-Palmer Theorem: *A* complex unital Banach algebra $(A, ||\cdot||)$ is a $B^*$-algebra if and only if $A = H(A, ||\cdot||) + iH(A, ||\cdot||)$, the involution being determined by sym $A = H(A, ||\cdot||)$. Here $H(A, ||\cdot||)$ denotes the set of hermition elements with real numerical range).

In [14], Giles and Koehler introduced numerical range $V(A, P, x)$ for an element x in a complete lmc algebra $A$ with a defining family $P$ of semi-norms. They establish the following Vidav-Palmer type characterization for $b^*$-algebras.

**Theorem 2.1.** Let $A$ be a complete unital lmc algebra with a defining $m$-calibration $P = (p_\alpha)$. Then the following are equivalent:

(a) $A$ is *a* $b^*$-algebra under some involution

(b) $A = H(A, P) + iH(A, P)$, a hermitian decomposition w.e.t. $P$.

(c) The subalgebra $B_p = \{x \in A \mid p(x) = \sup_\alpha p_\alpha(x) < \infty\}$ with the norm $p$ is a $B^*$-algebra under some involution and is dense in $A$.

In a subsequent paper [15] with Ioceph and Sims, they applied the theory to the study of quotient bounded operators on a loqally convex space. This gives them the following characterization of $b^*$-algebras.

**Theorem 2.2.** *A* $b^*$-algebra is topologically *isomorphic to a closed self-adjoint subalgebra of the $b^*$-algebra of all the quotient bounded operators on the product of a family of Hilbert spaces.

Wood [21] introduced numerical ranges for an element on locally convex algebra and for operators on locally convex spaces. His extended Vidav-Palmer theorem characterized $GB^*$-algebras as follows. *A* semi-$GB^*$-algebra is a $GB^*$-algebra without the continuity of the involution.

**Theorem 2.3.** Let $A$ be a complete hypocontinuous locally convex algebra. Then $A$ is a semi-$GB^*$-algebra (under some involution) if and only if there is a calibration for $A$ w.r.t. which $A$ has a hermitian decomposition.

This theorem of Wood has been used by Bhatt [5,6] to prove the part (b) of the following theorem.

**Theorem 2.4.** (*a*) Let $A$ be a $GB^*$-algebra with unit ball $B_0$. Then the $B^*$-algebra $A(B_0)$ is sequentially dense in $A$.

(*b*) Conversely, let $A$ be a complete hypocontinuous locally convex *algebra. If $A$ contains a *subalgebra $B$, with $1 \in B$, such that

(i) $B$ is a $B^*$-algebra under some norm $||.||$

(ii) $(B, ||\cdot||) \rightarrow A$ is a sequentially dense continuous injection, then $A$ is a $GB^*$-algebra.

## 3. MISCELLENIOUS RESULTS

Dixon's work lead to the study of the concrete unbounded operator algebras like extended $C^*$-algebras and extended $W^*$-algebras. An interesting result of Dixon [13] is the following:

**Theorem 3.1.** Let $A$ be a von Neumann algebra acting on a separable Hilbert space H.

**(a) The *algebra m $(A)$ of operators which are locally measurable w.r.t. $A$ with the topology of convergence locally** in measure is a $GB^*$-algebra with unit ball $B_0 = \{T \in A \mid ||T|| \leqslant 1\}$.

(b) Any extended W*-algbra of operators in $H$ over $A$ is a *subalgebra of m $(A)$.

Recently Inoue [16] has started investigating extended $C^*$- and extended $W^*$-algebras with common dense domain, which he called $EC^*$- algebras and $EW^*$-algebras. He has extended a number of $W^*$-algebra results to this setting; in particular, he has characterized weakly continuous and $\sigma$-weakly continuous positive functionals, obtained the structure theorem for $\sigma$-weakly continuous *homomorphisms between $EW^*$-algebras as well as has generalized von Neumann second commutant theorem.

We mention the following result of Inoue [17].

**Theorem 3.2.** Every representation of a $GB^*$-algebra $A$ with unit ball $B_0$ into closable Hiblert space operators is self-adjoint and its maps $A(B_0)$ into bounded operators.

This shows that when a $GB^*$-algebra $A$ acts on a Hilbert space H, the hermitian elements of $A$ are mapped into essentially self-adjoint operators, and not merely symmetric ones. This fact, together with Theorem 2.4 indicates the limitations of $GB^*$-model from the point of view of applications. There is another more suitable generalizatio of $B^*$-algebras due to Laßner [18]: Let H be a Hilbert space, $D$ a dense subspace of H, End $D$ be the algebra of all endomorphisms of D. Let $A$ be a subalgebra of End $D$ with the proprety that for each $T \in A$, the operator $T^+$ defined as $(Tx, y) = (X, T^+y)$ $(x,y \in D)$ is also in $A$. Then $A$ is a *algebra with involution $T \rightarrow T^+$. On $D$, a locally convex topology $t_A$ ts defined by the semi-norms $x \rightarrow ||Tx||$ $(T \in A)$. Let $G$ be the collection of all bounded subsets of $(D, t_A)$. For each $B \in G$, let

$$p_B(T) = \sup \{ | (Tx, y) | x, y \in B\}$$

and $T_D$ be the topology on $A$ defined by $\{p_B | B \in G\}$. Then $(A, T_D)$ is called an $O^*$-algebra; and any locally convex *algebra which is algebraically and topologically *isomorphic to an $\tilde{O}^*$-algebra is called an $\widetilde{AO}^*$-*algebra*. The following result due to Smudgen [20] is a nonspatial charcterization.

**Theorem 3.3.** Let $A$ be a barrelled locally convex *algebra. Then $A$ is an $\widetilde{AO}^*$-algebra if and only if the wedge $A^+$ of positive elements of $A$ is normal.

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## Р Е З И М Е

Оваа статија претставува краток преглед на настојувањата во последните дваесетина години за уопштување на $B^*$-алгебри. Ние се концентрираме само на два аспекта од теоријата: Структурните теореми и нумеричките области на вредности. Прегледот вклучува и некои најнови (се уште необјавени) резултати добиени од авторовиот студент Mr. S. Bhatt, за чија помош при изработката на оваа статија авторот искажува голема благодарност.