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ON $\big(k(n-1)+1\big)$–SEMIGROUPS WITH $(n-2)$–NEUTRAL OPERATIONS

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Abstract

In the present paper, we define [left, right] $(n-2)$–neutral operation E [: $Q^{n-2} \to Q$] of a $\big(k(n-1)+1\big)$–groupoid, $(k,n) \in N \times (N \setminus \{1\})$, so that (among others) for $n=2$ $\mathrm{E}(\emptyset)[a_1^{n-2}=\emptyset]$ is a neutral element of the $(k+1)$–groupoid (Q,A). The main result of the paper is the following proposition. If a $\big(k(n-1)+1\big)$–semigroup (Q,A), $k \geq 2$, has a left [right] $(n-2)$–neutral operation E, then there is an n–semigroup (Q,B) with $\{1,n\}$–neutral operation $\big[:[5],\ 1.2.2\big]$, such that for every $x_1^{k(n-1)+1} \in Q$, $A\left(x_1^{k(n-1)+1}\right) = \overset{k}{B}(x_1^{k(n-1)+1})$. [E.g.: $\overset{2}{B}(x_1^{2n-1}) \overset{\mathrm{def}}{=} B\big(B(x_1^n), x_{n+1}^{2n-1}\big)$.] Moreover, if $n \geq 3$ then (Q,A) is a $\big(k(n-1)+1\big)$–group.

1 Preliminaries.

1.1. About the expression $\overline{\overset{(i)}{a}{}_p^q}\Big|_{i=t}^{s}$

Let p and q be arbitrary natural numbers such that $p \leq q$, and t and s arbitrary element of the sets N and $N \cup \{0\}$, respectively. Further on, let $\overline{S}$ be the set of all sequence $\overset{(i)}{a}{}_p^q$ over a set $S(\emptyset \in S)$, and let

$$\overline{\overset{(i)}{a}{}_p^q}\Big|_{i=t}^{s} \tag{1}$$

and

$$c_p^{p+(s-t+1)(q-p+1)-1} \tag{2}$$

be arbitrary sequence the sets $\overline{S}$ and S, respectively. The sequence (1) is nonempty iff $t \leq s$. Moreover: the sequence (2) is nonempty iff $t \leq s$. $[t \leq s \Leftrightarrow (s-t+1)(q-p+1) \geq 1;\ q-p+1 \geq 1 \Leftrightarrow p \leq q]$. In addition, if $t \leq s$ and

$$(\forall i \in \{t,\dots,s\})(\forall j \in \{p,\dots,q\} c_{(i-t)(q-p+1)+j} = \overset{(i)}{a}_j\,, \tag{3}$$

then to every sequence (1) over $\overline{S}$ there corresponds exactly one sequence (2) over S, and conversely. Hence: if $t \leq s$ and (3) holds [since $\overline{\overset{(i)}{a}{}_p^q}\Big|_{i=t}^s = \emptyset \Leftrightarrow c_p^{p+(s-t+1)(q-p+1)-1} = \emptyset$], we use the **following convention**:

$$\overline{\overset{(i)}{a}{}_p^q}\Big|_{i=t}^s \quad \text{stands for} \quad c_p^{p+(s-t+1)(q-p+1)-1}\,.$$

1.2. About n–groups

1.2.1. Definitions: *Let $n \geq 2$ and let (Q,A) be an n–groupoid. Then:* (a) *we say that (Q,A) is an n–semigroup iff for every $i,j \in \{1,\dots,n\}$, $i < j$, the following law holds*

$$A\left(x_1^{i-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-1}\right) = A\left(x_1^{j-1},\ A(x_j^{j+n-1}),\ x_{j+n}^{2n-1}\right)$$

[*: $<i,j>$–associative law*]; (b) *we say that (Q,A) is an n–quasigroup iff for every $i \in \{1,\dots,n\}$ and for every $a_1^n \in Q$ is exactly one $x_i \in Q$ such that the following equality holds*

$$A(a_1^{i-1},\, x_i,\, a_i^{n-1}) = a_n\,; \quad \textit{and}$$

(c) *we say that (Q,A) is a Dörnte n–group [briefly: n–group] iff (Q,A) is an n–semigroup and an n–quasigroup as well.*

A notion of an n–group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

1.2.2. Definitions [5]: *Let $n \geq 2$ and let (Q,A) be an n–groupoid. Further on, let* **e** *be an mapping of the set Q^{n-2} into the set Q. Let also On $(k(n-1)+1)$–semigroups with $(n-2)$–neutral operations $\{i,j\} \subseteq \{1,\dots,n\}$ and $i < j$. Then:* **e** *is an $\{i,j\}$–neutral operation of the n–groupoid (Q,A) iff the following formula holds*

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)\big(A(a_1^{i-1},\ \mathbf{e}(a_1^{n-2}),\ a_i^{j-2},\ x,\ a_{j-1}^{n-2}) = x$$
$$\wedge A\big(a_1^{i-1},\ x,\ a_i^{j-2},\ \mathbf{e}(a_1^{n-2}),\ a_{j-1}^{n-2}\big) = x\big)\,.$$ [1]

1.2.3. Proposition [5]: *Let $n \geq 2$, $\{i,j\} \subseteq \{1,\dots,n\}$ and $i < j$. Then in every n–groupoid there is at most one $\{i,j\}$–neutral operation.*

1.2.4. Proposition [5]: *In every n–group, $n \geq 2$, there is a $\{1,n\}$–neutral operation.*[2]

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[1] For $n = 2$, $\mathbf{e}(a_1^{n-2})[=\mathbf{e}(\emptyset)] = e \in Q$ is a neutral element of the groupoid (Q,A).

[2] There are n–groups without $\{i,j\}$–neutral operations with $\{i,j\} \neq \{1,n\}$ [:[6]]. In [6], n–groups with $\{i,j\}$–neutral operations, for $\{i,j\} \neq \{1,n\}$ are described.

1.2.5. Proposition [5]: *Let $n \geq 3$ and let (Q, A) be an n–semigroup. Then: (Q, A) is an n–group iff (Q, A) has a $\{1, n\}$–neutral operation.* [3]

1.2.6. Remark: In [8] showed that the condition "...(Q, A) is an n–semigroup ..." can be weakened to the condition "...(Q, A) is an $< 1, 2 >$–associative n–groupoid ..." or to the condition "...(Q, A) is an $< n-1, n >$–associative n–groupoid ...".

1.2.7. Proposition [4, 7]: *Let $n \geq 3$ and let (Q, A) be an n–groupoid Further on, let* E *be an mapping of the set Q^{n-2} into the set Q. Then the following propositions are equivalent:*

(i) *(Q, A) is an n–group;*

(ii) *the laws $A\big(A(x_1^n), x_{n+1}^{2n-1}\big) = A\big(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}\big)$, $A\big(x, a_1^{n-2}, \mathrm{E}(a_1^{n-2})\big) = x$ and $A\big(b_1^{n-2}, \mathrm{E}(b_1^{n-2}), x\big) = x$ hold in algebra $(Q, \{A, \mathrm{E}\})$ of the type $< n, n-2 >$; and*

(iii) *the laws $A\big(x_1^{n-2}, A(x_{n-1}^{2n-1}), x_{2n-1}\big) = A\big(x_1^{n-1}, A(x_n^{2n-1})\big)$, $A\big(\mathrm{E}(a_1^{n-2}), a_1^{n-2}, x\big) = x$ and $A\big(x, \mathrm{E}(b_1^{n-2}), b_1^{n-2}\big) = x$ hold in algebra $(Q, \{A, \mathrm{E}\})$ of the type $< n, n-2 >$.* [4, 5]

1.3. On superpositions of an n–semigroup operation

1.3.1. Definition: *Let (Q, A) be an n–groupoid and $n \geq 2$. Then:*

1) $\overset{1}{B} \overset{\text{def}}{=} B$; *and*

2) *for every $k \in N$ and for every $x_1^{(k+1)(n-1)+1} \in Q$*

$$\overset{k+1}{B}\big(x_1^{(k+1)(n-1)+1}\big) \overset{\text{def}}{=} B\big(\overset{k}{B}\big(x_1^{k(n-1)+1}\big),\ x_{k(n-1)+2}^{(k+1)(n-1)+1}\big),$$

1.3.2. Proposition: *Let (Q, B) be an n-semigroup, $n \geq 2$ and $(i, j) \in N^2$. Then, for every $x_1^{(i+j)(n-1)+1} \in Q$ and for every $t \in \{1, \ldots, i(n-1)+1\}$, the following equality holds*

$$\overset{i+j}{B}\big(x_1^{(i+j)(n-1)+1}\big) = \overset{i}{B}\big(x_1^{t-1},\ \overset{j}{B}\big(x_t^{t+j(n-1)}\big),\ x_{t+j(n-1)+1}^{(i+j)(n-1)+1}\big).$$

An immediate consequence of Proposition 1.3.2. is the following proposition:

1.3.3. Proposition: *If (Q, B) is an n–semigroup [n–group], then* $(Q, \overset{k}{B})$ is a $\big(k(n-1)+1\big)$–semigroup $\big[\big(k(n-1)+1\big)$–group$\big]$.

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[3] This result has been commented from the particular point of view in the paper [8] [: Remark 5.2].

[4] See Corollary 5 in [4] and Theorem 2.6 in [7]. (The sketch of the proof of this proposition can be found in [9]).

[5] If (Q, A) is an n–group, then **E** its $\{1, n\}$–neutral operation [: 1.2.7, 1.2.1, 1.2.2, 1.2.4].

Remark: More about superpositions of an n–semigroup operation [with different notations] can be found in [3].

2. Results

2.1. Definition: *Let* $(k,n)\in N\times(N\setminus\{1\})$, *let* A *be a* $\big(k(n-1)+1\big)$ *-ary operation in* Q *and* E *a mapping of the set* Q^{n-2} *into the set* Q. *Then:*

1) *we say that* E *is a* **left** $(n-2)$**–neutral operation** *of a* $\big(k(n-1)+1\big)$ **-groupoid**. *On* $\big(k(n-1)+1\big)$*–semigroups with* $(n-2)$*–neutral operations* (Q,A) *iff the formula*

$$(\forall \overset{(1)}{a_t}\in Q)_1^{n-2}\dots(\forall \overset{(k)}{a_t}\in Q)_1^{n-2}$$
$$\Big(\bigwedge_{i=1}^{k} A\big(\overline{E(\overset{(j)}{a}{}_1^{n-2}),\ \overset{(j)}{a}{}_1^{n-2}}\,\big|_{j=1}^{i}\ x,\ \overline{\mathrm{E}(\overset{(j)}{a}{}_1^{n-2})\ \overset{(j)}{a}{}_1^{n-2}}\,\big|_{j=i+1}^{k}\big) = x\Big) \qquad (1)$$

holds;

2) *we say that* E *is a* **right** $(n-2)$**–neutral operation** *of a* $\big(k(n-1)+1\big)$-**grupoid** (Q,A) *iff the formula*

$$(\forall \overset{(1)}{a_t}\in Q)_1^{n-2}\dots(\forall \overset{(k)}{a_t}\in Q)_1^{n-2}$$
$$\Big(\bigwedge_{i=1}^{k} A\big(\overline{(\overset{(j)}{a}{}_1^{n-2}),\mathrm{E}(\overset{(j)}{a}{}_1^{n-2})}\,\big|_{j=1}^{i-1}\ x,\ \overline{\overset{(j)}{a}{}_1^{n-2}\ \mathrm{E}(\overset{(j)}{a}{}_1^{n-2})}\,\big|_{j=i}^{k}\big)= x\Big) \qquad (2)$$

holds; and

3) *we say that* E *is a* $(n-2)$**–neutral operation** *of a* $\big(k(n-1)+1\big)$-**groupoid** (Q,A) *iff* E *is a* **left** $(n-2)$*–neutral operation of a* $\big(k(n-1)+1\big)$*-groupoid* (Q,A) *and a* **right** $(n-2)$*–neutral operation of a* $\big(k(n-1)+1\big)$*-groupoid* (Q,A).

2.2. Remark: For $n=2$ the formula (1) and the formula (2) [from 2.1] reduces, respectively, to the following formulas

$$(\forall x\in Q)\Big(\bigwedge_{i=1}^{k} A(\overset{i}{e}, x, \overset{k-1}{e}) = x\Big) \qquad (\hat{1})$$

and

$$(\forall x\in Q)\Big(\bigwedge_{i=1}^{k} A(\overset{i-1}{e}, x, \overset{k-i+1}{e}) = x\Big); \qquad (\hat{2})$$

$e=\mathrm{E}(\emptyset)$. Further on, the conjuction of the statements $(\hat{1})$ and $(\hat{2})$ is equivalent with the statement

$$(\forall x\in Q)\Big(\bigwedge_{i=1}^{k+1} A(\overset{i-1}{e}, x, \overset{k-i+1}{e}) = x\Big). \qquad (\mathbf{e})$$

Finally: $e\in Q$ is a **neutral element** of the $(k+1)$-groupoid (Q,A), $k\in N$, iff the formula (**e**) holds. □

By Definition 2.1, and also by the definition of an $\{1,n\}$–neutral operation of n–grupoid [: 1.2.2], by Proposition 1.2.3 and finally by Remark 2.2, we conclude that the following proposition holds:

2.3. Proposition: *Let be a* $(Q,A)\big(k(n-1)+1\big)$*-groupoid,* E *its* $(n-2)$*-neutral operation and* $(k,n)\in N\times(N\setminus\{1\})$*. Then:*

1) *If* $k=1$*, then*

- E *is a* $\{1,n\}$*-neutral operation of the* n*-groupoid* (Q,A)*,*
- *for* $n=2$E$(\emptyset)$ *is a neutral element of the groupoid* (Q,A)*, and*
- E *is uniquely determined for every* $n\geq 2$*; and*

2) *If* $k\in N$ *and* $n=2$*. then* E$(\emptyset)$ *is a neutral element of a* $(k+1)$*-groupoid* (Q,A). □

By 1.2.5, 1.2.7, 1.3.1, 1.3.2, 1.3.3, 2.2 *and* 2.3, *we conclude that the following proposition holds:*

2.4. Proposition: *Let* $n\geq 2$ *and let* (Q,B) *be an* n*-semigroup with a* $\{1,n\}$*-neutral operation* **e** [: 1.2.2]. *Further on, let* $k\geq 2$*. Then the following statements hold:* a) **e** *is an* $(n-2)$*-neutral operation of a* $\big(k(n-1)+1\big)$*-semigroup* $(Q,\overset{k}{B})$*; and* b) *if* $n\geq 3$*, then* $(Q,\overset{k}{B})\big(k(n-1)+1\big)$*-group.*

2.5. Theorem: *Let* $k\geq 2$*,* $n\geq 2$ *and let* (Q,A) *be a* $\big(k(n-1)+1\big)$*-semigroup. Further on, let* E *be a* **left** $(n-2)$*-neutral operation of a* $\big(k(n-1)+1\big)$*-semigroup* (Q,A) **or** *a* **right** $(n-2)$*-neutral operation of a* $\big(k(n-1)+1\big)$*-semigroup* (Q,A)*. Then there* **exists** *an* n*-groupoid* (Q,B) *such that the following statements hold:*(i) (Q,B) *is an* n*-semigroup;* (ii) $A=\overset{k}{B}$*; and* (iii) E *is a* $\{1,n\}$*-neutral operation of the* n*-groupoid* (Q,B)*. Moreover, if* $n\geq 3$ *then* (Q,A) *is* $\big(k(n-1)+1\big)$*-group.*[6]

Proof. 1) Let E is a **right** $(n-2)$–neutral operation of a $\big(k(n-1)+1\big)$–semigroup (Q,A); $k\geq 2$, $n\geq 2$. We prove respectively that in that case the following statements hold.

1° Let $\overset{(j)}{a}{}_1^{n-2}, j\in\{1,\dots,n-1\}$, be arbitrary sequences over Q. Further On $\big(k(n-1)+1\big)$*-semigroups* with $(n-2)$=neutral operations on, let for every $x_1^n\in Q$

$$B(x_1^n)\overset{\text{def}}{=}A\big(x_1^n,\ \overline{\overset{(j)}{a}{}_1^{n-2},\mathrm{E}(\overset{(j)}{a}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big). \tag{a}$$

Then for every sequence of the sequences $\overset{(j)}{b}{}_1^{n-2}$, $j\in\{1,\dots,n-1\}$, over Q and for every $x_1^n\in Q$ the following equality holds

$$B(x_1^n)\overset{\text{def}}{=}A\big(x_1^n;\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big).$$

[6] The following proposition was proved in [2]. If (Q,A) is an m–semigroup, $m\geq 3$ and (Q,A) has a neutral element e, then there is semigroup $(Q,\cdot)$ with a neutral element such that, for every $x_1^m\in Q$, $A(x_1^m)=x_1\cdot\ \dots\ \cdot x_m$.

2° For every $x_1^n \in Q$, for every sequence of sequences $\overset{(j)}{B}{}_1^{n-1}$, $j \in \{1,\dots,n-1\}$, over Q and every $i \in \{1,\dots,n\}$ the following equality holds

$$B(x_1^n) = A\big(x_1^{i-1}, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}, x_i^n\big).$$

3° (Q,B), where the n-ary operation B in Q is defined by (a) in 1°, is an n-semigroup.

4° For every $x_1^{k(n-1)+1} \in Q$ the following equality holds

$$A(x_1^{k(n-1)+1}) = \overset{k}{B}(x_1^{k(n-1)+1}).$$

5° The right $(n-2)$-neutral operation E of the $\big(k(n-1)+1\big)$-semigroup (Q,A) is an $\{1,n\}$-neutral operation of the n-semigroup (Q,B). Moreover, if $n \geq 3$ then (Q,A) is a $\big(k(n-1)+1\big)$-group.

The proof of the statement 1°:

For every $x_1^n \in Q$, for every sequence $\overset{(k)}{a}{}_1^{n-2}$ over Q and for every sequence of sequences $\overset{(j)}{b}{}_1^{n-2}$, $j \in \{1,\dots,k-1\}$, over Q the following series of equalities holds

$$B(x_1^n) = A\big(B(x_1^n), \overset{(k)}{a}{}_1^{n-2}, \mathrm{E}(\overset{(k)}{a}{}_1^{n-2}), \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big) =$$

$$= A\big(A(x_1^n, \overline{\overset{(j)}{a}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{a}{}_1^{n-2})}\,\big|_{j=1}^{k-1}), \overset{(k)}{a}{}_1^{n-2}, \mathrm{E}(\overset{(k)}{a}{}_1^{n-2}), \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2}), \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big) =$$

$$= A\big(x_1^{n-1}, A\big(x_n, \overline{\overset{(j)}{a}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{a}{}_1^{n-2}),}\,\big|_{j=1}^{k}\big), \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big) =$$

$$= A\big(x_1^n, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big)$$ [: (2) from 2.1, (a) from 1°, (a) from 1.2.1].

The proof statement 2°:

Let i be an arbitrary element of the set $\{1,\dots,n\}$. Then for an arbitrary sequence x_1^n over Q, an arbitrary sequence of sequences $\overset{(j)}{b}{}_1^{n-2}$, $j \in \{1,\dots,k-1\}$, over Q and for arbitrary sequence c_1^{n-2} over Q the following series of equalities holds

$$A\big(x_1^i, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}, x_{i+1}^n\big) =$$

$$= A\big(x_1^{i-1}, A(\overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}, x_i, c_1^{n-2}, \mathrm{E}(c_1^{n-2})\big), \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}, x_{i+1}^n\big) =$$

$$= A\big(x_1^{i-1}, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}, A(x_i, c_1^{n-2}, \mathrm{E}(c_1^{n-2}), \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big), x_{i+1}^n\big) =$$

$$= A\big(x_1^{i-1}, \overline{\overset{(j)}{b}{}_1^{n-2}, \mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\big|_{j=1}^{k-1}\big), x_i^n\big)$$ [: (2) from 2.1. (a) from from 1.2.1].

The proof of the statement 3°:

Let i be an arbitrary element of the set $\{1,\dots,n-1\}$. Then for every $x_1^{2n-1}\in Q$, for every sequence of sequences $\overset{(j)}{b}{}_1^{n-2}$, $j\in\{1,\dots,k-1\}$, over Q and for every sequence of sequences $\overset{(j)}{c}{}_1^{n-2}$, $j\in\{1,\dots,k-1\}$, over Q the following series of equalities holds

$$A\big(x_1^{i-1},A\big(x_i^{i+n-1},\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}\big),x_{i+n}^{2n-1},\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathrm{E}(\overset{(j)}{c}{}_1^{n-2},)}\,\Big|_{j=1}^{k-1}\big)=$$

$$=A\big(x_1^{i},A\big(x_{i+1}^{i+n-1},\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}\big),x_{i+n}\big),x_{i+n+1}^{2n-1},\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathrm{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}\big).$$

On $(k(n-1)+1)$-semigroups with $(n-2)$-neutral operations [: (a) from 1.2.1], where, by 1° and 2°, we conclude that for every $x_1^{2n-1}\in Q$ the following equality holds

$$B\big(x_1^{i-1},\ B(x_i^{i+n-1}),x_{i+n}^{2n-1}\big)=B\big(x_1^{i},\ B(x_{i+1}^{i+n}),\ x_{i+n+1}^{2n-1}\big)$$

The proof of the statement 4°:

a) For every $x_1^{k(n-1)+1}\in Q$, for every sequence b_1^{n-2} over Q and for every sequence of sequences $\overset{(j)}{c}{}_1^{n-2}$, $j\in\{1,\dots,k-1\}$, over Q the following series of equalities holds

$$A\big(x_1^{(k-1)(n-1)},\ B(x_{(k-1)(n-1)+1}^{k(n-1)+1}),\ b_1^{n-2},\ \mathrm{E}(b_1^{n-2})\big)=$$

$$=A\big(x_1^{(k-1)(n-1)},A(x_{(k-1)(n-1)+1}^{k(n-1)+1}),\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathrm{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1}\big),b_1^{n-2},\mathrm{E}(b_1^{n-2})\big)=$$

$$=A\big(x_1^{k(n-1)},A(x_{k(n-1)+1}),\ \overline{\overset{(j)}{c}{}_1^{n-2},\mathrm{E}(\overset{(j)}{c}{}_1^{n-2})}\,\Big|_{j=1}^{k-1},b_1^{n-2},\mathrm{E}(b_1^{n-2})\big)=$$

$$=A(x_1^{k(n-1)+1})\ \ [:\ 1°,\ (2)\text{ from }2.1].$$

Whence, besides, we conclude that 4° for $k=2$ holds [: 1°, 1.3.2, (2) from 2.1].

b) Let $k>2$. Further on, let i be an arbitrary integer such that $i\le k-2[\Leftrightarrow k-(i+1)\ge 1]$. Then for every $x^{k(n-1)+1}\in Q$, for every sequence of sequences $\overset{(j)}{b}{}_1^{n-2}$, $j\in\{1,\dots,k-1\}$, over Q and for every sequence of sequences $\overset{(t)}{c}{}_1^{n-2}$, $t\in\{1,\dots,k-1\}$, over Q the following series of equalities holds

$$A\big(x_1^{i(n-1)},\ \overset{k-i}{B}\,(x_{i(n-1)+1}^{k(n-1)+1}),\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=i}^{k-1}\big)=$$

$$=A\big(x_1^{i(n-1)},B\big(x_{i(n-1)+1}^{(i+1)(n-1)},\ \overset{k-(i+1)}{B}\,(x_{(i+1)(n-1)+1}^{k(n-1)+1})\big),\ \overline{\overset{(j)}{b}{}_1^{n-2},\mathrm{E}(\overset{(j)}{b}{}_1^{n-2})}\,\Big|_{j=i}^{k-1}\big)=$$

$$= A\big(x_1^{i(n-1)}, A(x_{i(n-1)+1}^{(i+1)(n-1)}, \overset{k-(i+1)}{B} \times$$

$$\times (x_{(i+1)(n-1)+1}^{k(n-1)+1}), \overline{\overset{(t)}{c_1^{n-2}}, \mathrm{E}(\overset{(t)}{c_1^{n-2}})}\Big|_{t=1}^{k-1}), \overline{\overset{(j)}{b_1^{n-2}}, \mathrm{E}(\overset{(j)}{b_1^{n-2}})}\Big|_{j=i}^{k-1}\big) =$$

$$= A(x_1^{(i+1)(n-1)}, A(\overset{k-(i+1)}{B}(x_{(i+1)(n-1)+1}^{k(n-1)+1}), \overline{\overset{(t)}{c_1^{n-2}}, \mathrm{E}(\overset{(t)}{c_1^{n-2}})}\Big|_{t=1}^{k-1}), \overset{(i)}{b_1^{n-2}}, \mathrm{E} \times$$

$$\times(\overset{(i)}{b_1^{n-2}}), \overline{\overset{(j)}{b_1^{n-2}}, \mathrm{E}(\overset{(j)}{b_1^{n-2}})}\Big|_{j=i+1}^{k-1}) =$$

$= A(x_1^{(i+1)(n-1)}, B^{k-(i+1)}(x_{(i+1)(n-1)+1}^{k(n-1)+1}), \overline{\overset{(j)}{b_1^{n-2}}, \mathrm{E}(\overset{(j)}{b_1^{n-2}})}\Big|_{j=i+1}^{k-1})$ [: 1.3.2, 1°, (2) from 2.1].

The proof of the statement 5°:

By (2) from 2.1, we conclude that for every $x \in Q$, for every sequence a_1^{n-2} over Q and for every sequence of sequences $\overset{(j)}{c_1^{n-2}}$, $j \in \{1, \ldots, k-1\}$, over Q the following series of equalities holds

$$A\big(x, a_1^{n-2}, \mathrm{E}(a_1^{n-2}), \overline{\overset{(j)}{c_1^{n-2}}, \mathrm{E}(\overset{(j)}{c_1^{n-2}})}\Big|_{j=1}^{k-1}\big) = x$$

and

$$A\big(a_1^{n-2}, \mathrm{E}(a_1^{n-2}), x, \overline{\overset{(j)}{c_1^{n-2}}, \mathrm{E}(\overset{(j)}{c_1^{n-2}})}\Big|_{j=1}^{k-1}\big) = x,$$

whence, by 1°, we conclude that the following formula is satisfied

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)\big(B(x, a_1^{n-2}, \mathrm{E}(a_1^{n-2})) = $$
$$= x \wedge B\big(a_1^{n-2}, \mathrm{E}(a_1^{n-2}), x\big) = x\big).$$

For $n = 2$ this formula reduces to the formula

$$(\forall x \in Q)\big(B(x, \mathrm{E}(\emptyset)) = x \wedge B\big(\mathrm{E}(\emptyset), x\big) = x\big)$$

[: 1.2.2, foot-note 1)], and for $n \geq 3$, by Proposition 1.2.7, Proposition 1.2.4 and 1.2.1, we conclude the following statement holds: E is an $\{1, n\}$–neutral operation of the n–semigroups (Q, B). Whence, by 1.3.3 (and by 1.2.5), we conclude that (Q, A) is a $\big(k(n-1)+1\big)$–group for $n \geq 3$.

2) Let E be a **left** $(n-2)$–neutral operation of the $\big(k(n-1)+1\big)$–semigroup (Q, A); $k \geq 2$, $n \geq 2$. Then by a simple imitation of the proof of statements 1°–5° it is possible to prove that following statements hold.

°1 Let $\overset{(j)}{a_1^{n-2}}$, $j \in \{1, \ldots, k-1\}$, be arbitary sequences over Q. Further on, let for every $x_1^n \in Q$

$$(\overline{a})\overline{B}(x_1^n) \overset{\mathrm{def}}{=} A\big(\overline{\mathrm{E}(\overset{(j)}{a_1^{n-2}}), \overset{(j)}{a_1^{n-2}}}\Big|_{j=1}^{k-1}, x_1^n\big).$$

On $\big(k(n-1)+1\big)$–semigroups with $(n-2)$–neutral operations.

Then for every sequence of sequences $\overset{(j)}{b_1^{n-2}}$, $j \in \{1, \ldots, k-1\}$, over Q and for every $x_1^n \in Q$ the following equality holds

$$\overline{B}(x_1^n) = A\big(\overline{\mathrm{E}(\overset{(j)}{b}{}_1^{n-2}),\ \overset{(j)}{b}{}_1^{n-2}}\,\big|_{j=1}^{k-1}, x_1^n\big).$$

°2 For every $x_1^n \in Q$, for every sequence of sequencees $\overset{(j)}{b}{}_1^{n-2}$, $j \in \{1,\dots,k-1\}$, over Q and for every $i \in \{1,\dots,n\}$ the following equality holds

$$\overline{B}(x_1^n) = A\big(x_1^i, \overline{\mathrm{E}(\overset{(j)}{b}{}_1^{n-2}),\ \overset{(j)}{b}{}_1^{n-2}}\,\big|_{j=1}^{k-1}, x_{x+1}^n\big).$$

°3 $(Q,\overline{B})$ is an n–semigroup, where $\overline{B}$ is an n–ary operation in Q defined by $(\overline{a})$ in °1.

°4 For every $x_1^{k(n-1)+1} \in Q$ the following equality holds

$$A(x_1^{k(n-1)+1}) = \overset{k}{\overline{B}}\,(x_1^{k(n-1)+1}).$$

°5 A left $(n-2)$–neutral operation E of the $\big(k(n-1)+1\big)$–semigroup (Q,A) is an $\{1,n\}$–neutral operation of the n–semigroup $(Q,\overline{B})$. Moreover, if $n \geq 3$, then (Q,A) is a $\big(k(n-1)+1\big)$–group.

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ЗА $(k(n-1)+1)$-ПОЛУГРУПИ СО $(n-2)$–НЕУТРАЛНИ ОПЕРАЦИИ

Јанез Ушан

Р е з и м е

Во трудов дефинираме (лева, десна) $(n-2)$-неутрална операција $\mathrm{E}[: Q^{n-2} \to Q]$ на еден $\big(k(n-1)+1\big)$-групоид, $(k,n) \in N \times (N \setminus \{1\})$, така што за $n=2$ $\mathrm{E}(\emptyset)[a_1^{n-2}=\emptyset]$ е неутрален елемент на $(k+1)$ групоидот (Q,A). Главниот резултат е следниов: Ако $\big(k(n-1)+1\big)$-полугрупата (Q,A), $k \geq 2$, има лева (десна) $(n-2)$-неутрална операција E, тогаш постои n-полугрупата (Q,B) со $\{1,n\}$-неутрална операција $\big[:[5], 1.2.2\big]$, така што за секои $x_1^{k(n-1)+1} \in Q$, $A\left(x_1^{k(n-1)+1}\right) = \overset{k}{B}(x_1^{k(n-1)+1})$. [E.g.: $\overset{2}{B}(x_1^{2n-1}) \overset{\mathrm{def}}{=} B\big(B(x_1^n), x_{n+1}^{2n-1}\big)$]. Уште повеќе, ако $n \geq 3$, тогаш (Q,A) е $\big(k(n-1)+1\big)$-група.

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