Математички Билтен
27 (LIII)
2003 (5–18)
Скопје, Македонија

ISSN 0351–336X

FORMAL FROBENIUS STRUCTURES GENERATED BY GEOMETRIC DEFORMATION ALGEBRAS

Mircea Crâşmăreanu

Abstract

Necessary and sufficient conditions for some deformation algebras to provide formal Frobenius structures are given. Also, examples of formal Frobenius structures with fundamental tensor that is not of the deformation type and examples of symmetric non-metric connections are presented.

Introduction

Frobenius manifolds were introduced by Dubrovin ([5]) as a "coordinate-free" framework for Gromov-Witten invariants. Also, Frobenius manifolds provide a natural geometric setting for understand the bi-Hamiltonian structure of hydrodynamics systems. The present paper is devoted to a notion related to Frobenius manifolds, namely *formal Frobenius structure.*

Our aim is to connect the geometry of formal Frobenius structures to classical differential geometry through deformation algebras of connections, a notion introduced by Izu Vaisman in [21]. The starting point of our study is the following remark: a formal Frobenius structure is a pair (g, A) with g a Riemann metric and A a tensor field of $(1, 2)$-type, subject to the conditions below. Naturally associated to g is the Levi-Civita connection ∇ and then the pair (∇, A) yields another linear connection $\overline{\nabla} = \nabla + A$. We search the converse, namely starting with $(\nabla, \overline{\nabla})$ we find conditions for $A = \overline{\nabla} - \nabla$ to satisfy the definition of a Frobenius structure. Therefore, in a general sense, every Frobenius structure can be viewed as a geometric deformation algebra.

An usual condition in the theory of Frobenius manifolds is the symmetry of A which is equivalent in our framework with the equality of torsions for ∇ and $\overline{\nabla}$. Because ∇ is without torsion we arrive at condition of torsionless of $\overline{\nabla}$ which restricts the number of remarkable deformation algebras. So, we add a version of Frobenius structures without commutativity condition.

Let us sketch the contents: after a first section which reviews the main definitions, in the following two sections several examples of formal and weak Frobenius structures provided by deformation algebras are discussed. Another section is devoted to some examples of Frobenius structures with A not of $\overline{\nabla}-\nabla$ type. Because in the above discussion the search of torsion-free linear connections appears as very important, the last section gives examples of symmetric connections which are not Levi-Civita connection for any Riemannian metric.

1. Frobenius structures and deformation algebras

Let (M,g) be a fixed Riemannian manifold for which we denote $C^{\infty}(M)$ the ring of smooth real functions, $\mathcal{X}(M)$ the Lie algebra of vector fields, $T^r_s(M)$ the $C^{\infty}(M)$-module of tensor fields of (r,s)-type, $\Omega^k(M)$ the $C^{\infty}(M)$-module of k-differential forms. Let $n=\dim M$ be finite.

Definition 1.1. ([p. 286, 18]) *The triple (M,g,A) with $A\in T^1_2(M)$ is called a formal Frobenius structure if:*
(i) *for every $X,Y,Z\in\mathcal{X}(M)$:*

$$g(A(X,Y),Z)=g(X,A(Y,Z)) \tag{1.1}$$

(ii) *A is commutative i.e. $A(X,Y)=A(Y,X)$.*

Using the symmetry of g it follows that (1.1) means the invariance of application $g(A(\cdot,\cdot),\cdot):\mathcal{X}(M)^3\to\mathcal{X}(M)$ with respect to cyclic permutations.

Recall that for the given $A\in T^1_2(M)$ the multiplication $X*Y:=A(X,Y)$ defines a $C^{\infty}(M)$-algebra structure on $\mathcal{X}(M)$. Sometimes in the definition of formal Frobenius structures one also asks for a third condition, namely the existence of a unit element in this algebra but we do not work in this framework. For other several types of Frobenius structures see [9], [11], [19].

Given two linear connections $\nabla,\overline{\nabla}$ the $C^{\infty}(M)$-algebra defined by $A=\overline{\nabla}-\nabla$ is called *the deformation algebra* in [21 p. 83] (see also [2]). In the cited paper it is proved that the deformation algebra is commutative if and only if ∇ and $\overline{\nabla}$ have the same torsion. If we start with ∇ the Levi-Civita connection of g it follows that $\overline{\nabla}$ must be symmetric (torsionless). This condition is very restrictive and then, in order to use some remarkable deformation algebras which are not commutative, we consider:

Definition 1.2. *A triple satisfying only condition* (i) *of Definition* 1.1 *is called a weak Frobenius structure.*

2. Frobenius structures generated by deformation algebras

Example 2.1. Subgeodesic correspondences

Two Riemannian metrics $g, \overline{g}$ are said *in a g-subgeodesic correspondence* if there exists $\theta \in \Omega^1(M)$ and $P \in \mathcal{X}(M)$ such that $A = \overline{\nabla} - \nabla$ is given by:

$$A = \theta \otimes \delta + \delta \otimes \theta + g \otimes P \tag{2.1}$$

where $\nabla, \overline{\nabla}$ is the Levi-Civita connection of $g, \overline{g}$ and δ is the Kronecker tensor. Let $\psi \in \Omega^1(M)$ be the g-dual of P i.e. $g(P, X) = \psi(X)$ for every $X \in \mathcal{X}(M)$. A straightforward computation gives:

Proposition 2.1. *The triple* $(M, g, \overline{g})$ *yields a Frobenius structure if and only if for every* $X, Y, Z \in \mathcal{X}(M)$:

$$\theta(X) g(Y, Z) + \psi(Z) g(Y, X) = \theta(Z) g(Y, X) + \psi(X) g(Y, Z). \tag{2.2}$$

Let us consider the particular case when θ and ψ are proportional i.e. there exists $f \in C^\infty(M)$ such that $\psi = f\theta$. Relation (2.2) becomes:

$$(1 - f)\theta(X) g(Y, Z) = (1 - f)\theta(Z) g(Y, X). \tag{2.3}$$

Proposition 2.2. *Suppose that* $f \neq 1$, M *is connected with* $n \geq 2$ *and the triple* $(M, g, \overline{g})$ *yields a Frobenius structure. Then* $\theta = 0 = \psi, P = 0$ *and the Frobenius structure is degenerate i.e.* $A = 0$.

Proof. Let $\{U_i\}_{1 \leq i \leq n}$ be a g-orthonormal basis for $\mathcal{X}(M)$. For $X = U_i$, $Y = Z = U_j, i \neq j$, (2.3) reads $\theta(U_i) = 0$. But i is arbitrary. □

Particular cases:

1) If $P = 0$ then $g, \overline{g}$ are in *a geodesic* (or *projective*) correspondence. By a well-known result of Weyl, in this case g and $\overline{g}$ have the same geodesics.

2) For a *conformal change* $\overline{g} = e^{2u}g$ with $u \in C^\infty(M)$, we have (2.1) with $\theta = du$ (the differential of u) and $P = -\nabla u$ (the gradient of u).

Application: Two-dimensional Einstein spaces

Let (M, g) be an Einstein space with R the Ricci tensor; hence $R = \lambda g$ with $\lambda \in C^\infty(M) \setminus \{0\}$. For $n \geq 3$ λ is a constant and then we confine ourselves to the two-dimensional case. Suppose that R is nondegenerate and let $\overline{\nabla}$ be the Levi-Civita connection of R. Supposing that A yields a Frobenius structure we get that λ is constant and $A = 0$. □

In conclusion, returning to the general framework, the only favorable case (2.1) is when θ is exactly the g-dual of P. Using a standard notation in Riemannian geometry the relation (2.1) reads:

$$A = \theta \otimes \delta + \delta \otimes \theta + g \otimes \theta^{\#}. \tag{2.1'}$$

Example 2.2. Hypersurfaces with nondegenerate second fundamental form

Let M be a hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$ and g, b the first and second fundamental form of M. Suppose that $rank\ b = n$ let $\nabla, \overline{\nabla}$ be the Levi-Civita connection of g, b. The corresponding A it given by ([17]):

$$b\left(A\left(X,Y\right),Z\right) = -\frac{1}{2}\left(\nabla_X b\right)\left(Y,Z\right) \tag{2.4}$$

which yields the Frobenius relation

$$\left(\nabla_X b\right)\left(Y,Z\right) = \left(\nabla_Y b\right)\left(Z,X\right). \tag{2.5}$$

But this is exactly the Codazzi equation and then:

Proposition 2.3. *The triple (M, b, A) is a formal Frobenius structure.*

Example 2.3. Riemannian metrics related by a self-adjoint operator

Let $g, \widetilde{g}$ be two Riemannian metrics on M. Then there exists a unique $J \in T_1^1(M)$ such that:

$$\widetilde{g}\left(X,Y\right) = g\left(X,JY\right) = g\left(JX,Y\right) \tag{2.6}$$

for every $X, Y, Z \in \mathcal{X}(M)$. If $\nabla, \widetilde{\nabla}$ is the Levi-Civita connection of $g, \widetilde{g}$ then:

$$\begin{aligned}\widetilde{g}\left(A\left(X,Y\right),Z\right) = g\left(X,\left(\nabla_Y J\right)Z - \left(\nabla_Z J\right)X\right) + g\left(Y,\left(\nabla_X J\right)Z - \left(\nabla_Z J\right)X\right)\\ + g\left(Z,\left(\nabla_X J\right)Y + \left(\nabla_Y J\right)X\right)\end{aligned} \tag{2.7}$$

where, as usual, $A = \widetilde{\nabla} - \nabla$. Then we get:

Proposition 2.4. *The triple $(M, \widetilde{g}, A)$ is a Frobenius structure if and only if:*

$$g\left(Y,\left(\nabla_X J\right)Z - \left(\nabla_Z J\right)X\right) = g\left(X,\left(\nabla_Z J\right)Y\right) - g\left(Z,\left(\nabla_X J\right)Y\right). \tag{2.8}$$

Particular case

Let us suppose that J is ∇-*recurrent* i.e. there exists $\omega \in \Omega^1(M)$ such that $\nabla_X J = \omega(X) J$. Then:

Proposition 2.5. *With above condition the triple $(M, \widetilde{g}, A)$ is a Frobenius structure if and only if:*

$$\omega(X) g(Y, Z) = \omega(Z) g(Y, X). \tag{2.9}$$

3. Weak Frobenius structures generated by deformation algebras

Example 3.1 Golab connections Let $\theta \in \Omega^1(M)$ and $F \in T_1^1(M)$.

Definition 3.1. A linear connection $\overline{\nabla}$ associated to the triple (g, θ, F) is called a *Golab connection* if ([8]):

(i) it is a metric connection i.e. $\overline{\nabla}_X g = 0$

(ii) it has the torsion:

$$\overline{T}(X, Y) = \theta(Y) F(X) - \theta(X) F(Y). \tag{3.1}$$

The Golab connection exists and is unique, with expression ([13]):

$$\overline{\nabla}_X Y = \nabla_X Y + \theta(Y) F(X) - S(X, Y) P \tag{3.2}$$

where $S(X, Y) = g(FX, Y)$ and P is the g-dual of θ i.e. $g(P, X) = \theta(X)$ for every $X \in \mathcal{X}(M)$. It follows:

$$A(X, Y) = \theta(Y) F(X) - S(X, Y) P$$

and a straightforward computation gives:

Proposition 3.2. *The Golab triple (g, θ, F) yields a weak Frobenius structure if and only if for every $X, Y, Z \in \mathcal{X}(M)$:*

$$\theta(X) S(Y, Z) + \theta(Y) S(X, Z) = \theta(Z) [S(X, Y) + S(Y, X)]. \tag{3.3}$$

Application: λ-Hermitian metrics with respect to a ε-structure

rm

Let $\lambda, \varepsilon \in \mathbb{R} \backslash \{0\}$. F is called $\varepsilon-$structure if $F^2 = \varepsilon 1_{\mathcal{X}(M)}$ and g is called λ-Hermitian w.r.t. F if:

$$g(FX, FY) = \lambda g(X, Y). \tag{3.4}$$

Proposition 3.3. *If g is $\lambda-$Hermitian w.r.t. ε-structure F and (g,θ,F) yields a weak Frobenius structure then $\theta=0$ and therefore $A=0$.*

Proof. Obviously we can restrict to cases $\varepsilon,\lambda=\pm1$ and $\theta(P)=0$ implies $\theta=0$ because $\theta(P)=g(P,P)$ and then $P=0$. With $Z=P$ in (3.3) we get:

$$\theta(X)\theta(FY)+\theta(Y)\theta(FX)=\theta(P)\left[g(FX,Y)+g(X,FY)\right] \tag{3.5}$$

and $Y\to FY$ in (3.5) yields:

$$\varepsilon\theta(X)\theta(Y)+\theta(FX)\theta(FY)=\theta(P)g(X,Y)(\varepsilon+\lambda). \tag{3.6}$$

Again $Y=P$ in (3.6) leads to:

$$\theta(FX)\theta(FP)=\lambda\theta(X)\theta(P). \tag{3.7}$$

With $X=P$ in (3.7):

$$\theta(FP)^2=\lambda\theta(P)^2 \tag{3.8}$$

and then we have the cases:

I) $\lambda=-1$ implies $\theta(P)=0$.

II) $\lambda=+1$. Relations (3.7), (3.8) with $\lambda=1$ implies $\theta(FX)=\mu\theta(X)$ with $\mu=\pm1$ for every $X\in\mathcal{X}(M)$. Plugging in (3.6):

$$(\varepsilon+1)\theta(X)\theta(Y)=\theta(P)g(X,Y)(\varepsilon+1) \tag{3.9}$$

and we have the subclasses:

1) $\varepsilon=-1$. From (3.4) with $\lambda=1$ and $Y\to FY$ we get $-g(FX,Y)=g(X,FY)$ and thus (3.5) reads $2\mu\theta(X)\theta(Y)=0$ for every $X,Y\in\mathcal{X}(M)$ i.e. $\theta=0$.

2) $\varepsilon=+1$. Relation (3.9) becomes:

$$\theta(X)\theta(Y)=\theta(P)g(X,Y) \tag{3.10}$$

for every $X,Y\in\mathcal{X}(M)$. Let $\{U_i\}_{1\leq i\leq n}$ be an g-orthonormal basis for $\mathcal{X}(M)$. The choice $X=U_i, Y=U_j, i\neq j$, in (3.10) implies $\theta(U_i)\theta(U_j)=0$ and suppose $\theta(U_i)=0$. With $X=Y=U_i$ in (3.10) we have $\theta(U_i)^2=0=\theta(P)$. □

Remarks

(i) $\varepsilon=-1,\lambda=+1\,(-1)$ means that g is Hermitian (anti-Hermitian) w.r.t. *almost complex structure* F.

(ii) $\varepsilon=1,\lambda=-1$ means that g is para-Hermitian w.r.t. *almost product structure* F (p. 91, [3]).

(iii) An example for $\varepsilon = \lambda = 1$ is given by $F = 1_{\mathcal{X}(M)}$. The Golab connection for $\left(\theta, 1_{\mathcal{X}(M)}\right)$ is called *Lyra connection* ([10]).

Example 3.2. Cartan-Schouten connections on Lie groups
Let $M = G$ be a Lie group and $\{E_i\}_{1\leq i\leq n}$ a basis in $L(G)$ the Lie algebra of G. The *Cartan-Schouten connections* $\overline{\nabla}, \overset{+}{\nabla}, \overset{\circ}{\nabla}$ on G are:

$$\overline{\nabla}_{E_i}E_j = 0, \quad \overset{+}{\nabla}_{E_i} E_j = [E_i, E_j], \quad \overset{\circ}{\nabla}_{E_i} E_j = \frac{1}{2}[E_i, E_j] \tag{3.11}$$

Let g be the Riemannian metric with respect to which the given basis is orthonormal: $g(E_i, E_j) = \delta_{ij}$ and let ∇ the Levi-Civita connection of g. We define:

$$\overline{A}=\overline{\nabla}- \overset{\circ}{\nabla}, \overset{+}{A}=\overset{+}{\nabla} - \overset{\circ}{\nabla}, A=\overset{+}{\nabla}-\overline{\nabla}, \overset{+}{A}{}'=\overset{+}{\nabla}-\nabla, A'=\overset{\circ}{\nabla}-\nabla, \overline{A}'=\overline{\nabla}-\nabla \tag{3.12}$$

which yields (p. 35 [14]):

$$\begin{cases} A(E_i, E_j) = 2\overset{+}{A}(E_i, E_j) = -2\overline{A}(E_i, E_j) = [E_i, E_j] \\ 2g(\overline{A}'(E_i, E_j), E_k) = g(E_j, [E_i, E_k]) - g(E_k, [E_i, E_j]) + g(E_i, [E_j, E_k]) \\ 2g(\overset{+}{A}{}'(E_i, E_j), E_k) = g(E_i, [E_j, E_k]) + g(E_j, [E_i, E_k]) + g(E_k, [E_i, E_j]) \\ 2g(A'(E_i, E_j), E_k) = g(E_i, [E_j, E_k]) + g(E_j, [E_k, E_i]) \end{cases} \tag{3.13}$$

and then:

Proposition 3.4. *The Cartan-Schouten triples* $(G, g, A), \left(G, g, \overset{+}{A}\right), (G, g, \overline{A}), \left(G, g, \overline{A}'\right), \left(G, g, \overset{+}{A}{}'\right), (G, g, A')$ *are weak Frobenius structures if and only if:*

$$g(E_i, [E_j, E_k]) = g(E_j, [E_k, E_i]). \tag{3.14}$$

Remarks 3.5. (i) The torsions of the Cartan-Schouten connections are:

$$\overline{T}(X, Y) = -[X, Y], \quad \overset{+}{T}(X, Y) = [X, Y], \quad \overset{\circ}{T}(X, Y) = 0 \tag{3.15}$$

and then A' yields exactly a Frobenius structure if (3.14) holds.

(ii) A pair $(\mathbf{g}, <,>)$ with $(\mathbf{g}, [,])$ a Lie algebra and $<,>$ a scalar product on $\mathbf{g}$ such that:

$$<[x, y], z> + < y, [x, z] >= 0 \tag{3.16}$$

for every $x, y, z \in \mathbf{g}$ is called *orthogonal Lie algebra*. On a basis $\{E_i\}$ of $\mathbf{g}$ the relation (3.16) reads exactly as (3.14) and so we can restate the last proposition:

Proposition 3.4′. *The Cartan-Schouten triples are weak Frobenius structures if and only if* $(\mathcal{X}(G), [,])$ *is an orthogonal Lie algebra.* Recall also:

Proposition 3.6. *The following are equivalent:*
1) *the* A*-algebra is commutative*
2) *the* $\overset{+}{A}$*-algebra is commutative*
3) *the* $\overline{A}$*-algebra is commutative*
4) *the* $\overline{A}'$*-algebra is commutative*
5) *the Lie group* G *is abelian*
6) *the Lie algebra* $L(G)$ *is commutative.*

Therefore if G is abelian and (3.14) holds then $A, \overset{+}{A}, \overline{A}, \overline{A}'$ yields Frobenius structures.

Example 3.3. Complex connections on Kähler manifolds

Let (M, g, J) be a Kähler manifold. Recall that a linear connection $\overline{\nabla}$ on M is called *complex connection* if $\overline{\nabla} J = 0$ and the family of linear connections is given by:

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J) JY + \frac{1}{2}(Q(X,Y) - JQ(X,JY)) \qquad (3.17)$$

where ∇ is an arbitrary linear connection and also $Q \in T^1_2(M)$ is arbitrary.

Set ∇ the Levi-Civita connection of g; it results that $\overline{\nabla}$ is a complex connection because (M, g, J) is Kähler. Therefore $A = \overline{\nabla} - \nabla$ is:

$$A(X,Y) = \frac{1}{2}(Q(X,Y) - JQ(X,JY)) \qquad (3.18)$$

and then:

Proposition 3.7. *The Kähler triple* (M, g, J) *yields a weak Frobenius structure via* $Q \in T^1_2(M)$ *if and only if* Q *satisfy for every* $X, Y, Z \in \mathcal{X}(M)$:

$$g(Q(X,Y), Z) - g(Q(Y,Z), X) = g(Q(Y,JZ), JX) - g(Q(X,JY), JZ). \qquad (3.19)$$

Example 3.4. Chern and Bismut connections on Hermitian manifolds

Let (M, g, J) be a $2n$-dimensional $(n > 1)$ Hermitian manifold with complex structure J and compatible metric g. Let $\Omega \in \Omega^2(M)$ be *the*

Kähler form, $\Omega(\cdot,\cdot) = g(\cdot, J\cdot)$, and $\theta \in \Omega^1(M)$ *the Lee form*, $\theta = \frac{1}{n-1} d^*\Omega \circ J$. Let ∇ be the Levi-Civita connection of g and ∇^C, ∇^B the *Chern*, respectively *Bismut*, *connection* on (M, g, J):

$$\begin{cases} g\left(\nabla^C_X Y, Z\right) = g\left(\nabla_X Y, Z\right) + \frac{1}{2} d\Omega\left(JX, Y, Z\right) \\ g\left(\nabla^B_X Y, Z\right) = g\left(\nabla_X Y, Z\right) - \frac{1}{2} d\Omega\left(JX, JY, JZ\right). \end{cases} \tag{3.20}$$

For remarkable properties of these connections see [7].

Therefore, denoting $A^C = \nabla^C - \nabla, A^B = \nabla^B - \nabla$ the Chern, respectively Bismut, deformation tensor it results:

$$\begin{cases} g\left(A^C(X,Y), Z\right) = \frac{1}{2} d\Omega\left(JX, Y, Z\right) \\ g\left(A^B(X,Y), Z\right) = -\frac{1}{2} d\Omega\left(JX, JY, JZ\right) \end{cases} \tag{3.21}$$

and a straightforward computation gives:

Proposition 3.8. (i) *The triple* $\left(M, g, A^C\right)$ *is a weak Frobenius structure if and only if*:

$$d\Omega\left(JX, Y, Z\right) = d\Omega\left(JY, Z, X\right). \tag{3.22}$$

(ii) *The triple* $\left(M, g, A^B\right)$ *is a weak Frobenius structure if and only if the 3-form* $d\Omega\left(J\cdot, J\cdot, J\cdot\right)$ *is invariant to circular permutations i.e.*

$$d\Omega\left(JX, JY, JZ\right) = d\Omega\left(JY, JZ, JX\right). \tag{3.23}$$

Particular case $d\Omega = \theta \wedge \Omega$

For $n = 2$ i.e. in the 4-dimensional case or $n > 2$ and (M, g, J) is *locally conformal Kähler* (another notion introduced by I. Vaisman, [22]) we have $d\Omega = \theta \wedge \Omega$, [4]. Then we obtain:

Proposition 3.9. *Let* (M, g, J) *as above.*

(i) The triple $\left(M, g, A^C\right)$ *is a weak Frobenius structure if and only if*:

$$\begin{aligned} & d^*\Omega(X)\,\Omega(Y,Z) + d^*\Omega(JY)\,g(Z,X) - d^*\Omega(JZ)\,g(X,Y) = \\ & = d^*\Omega(Y)\,\Omega(Z,X) + d^*\Omega(JZ)\,g(X,Y) - d^*\Omega(JX)\,g(Y,Z). \end{aligned} \tag{3.24}$$

(ii) *The triple* $\left(M, g, A^B\right)$ *is a weak Frobenius structure if and only if the 3-form* $d^*\Omega(\cdot)\,\Omega(\cdot,\cdot)$ *is invariant to circular permutations*:

$$\begin{aligned} & d^*\Omega(X)\,\Omega(Y,Z) + d^*\Omega(Y)\,\Omega(Z,X) + d^*\Omega(Z)\,\Omega(X,Y) = \\ & = d^*\Omega(Y)\,\Omega(Z,X) + d^*\Omega(Z)\,\Omega(X,Y) + d^*\Omega(X)\,\Omega(Y,Z). \end{aligned} \tag{3.25}$$

4. (Weak) Frobenius structures with A not of $(\overline{\nabla} - \nabla)$-type

"Pseudo-"Example 4.1. Cross products on $I\!R^3$ and $I\!R^7$

A straightforward example of skew-symmetric weak Frobenius structure is given by: $M = I\!R^3$ or $I\!R^7, g = \langle \cdot , \cdot \rangle$ the Euclidean inner product and $A = \times$ the usual cross product, because for every $x, y, z \in I\!R^3$ we have $\langle x, y \times z\rangle = \langle y, z \times x\rangle = \langle z, x \times y\rangle$. In $I\!R^3$ this equality is well-known and for $I\!R^7$ see, for example, (p. 71, [15]). Still, we emphasize that the cross product is a pseudo-tensor and not a tensor.

Example 4.2. Frobenius structures generated by selfadjoint operators

Fix $J \in T_1^1(M)$ which is *g-selfadjoint* i.e. for every $X, Y \in X(M)$:

$$g(JX, Y) = g(X, JY). \tag{4.1}$$

Proposition 4.1. *If J is self-adjoint and for every $X, Y \in X(M)$:*

$$(\nabla_X J)Y = (\nabla_Y J)X \tag{4.2}$$

then (M, g, A) with $A(X, Y) = (\nabla_X J)Y$ is a Frobenius structure.

Proof. Let us apply ∇_X to (4.1) written in Y and Z:

$$\nabla_X(g(JY, Z)) = \nabla_X(g(Y, JZ))$$

which yields:

$$g(\nabla_X JY, Z) + g(Y, J(\nabla_X Z)) = g(\nabla_X JZ, Y) + g(J(\nabla_X Y), Z)$$

or:

$$g(\nabla_X JY - J(\nabla_X Y), Z) = g(Y, \nabla_X JZ - J(\nabla_X Z))$$

which means:

$$g((\nabla_X J)Y, Z) = g(Y, (\nabla_X J)Z) \overset{(4.2)}{=} g(Y, (\nabla_Z J)X)$$

a relation equivalent with Frobenius condition (1.1). Also, (4.2) means the commutativity of A. □

Particular cases:

1) Let (M, g) be an *invariant* hypersurface of a Riemannian manifold i.e. the curvature tensor of M is tangent to M. Let J be the Weingarten operator of M. Then (4.2) is exactly the Codazzi equation and (4.1) holds because:

$$g(JX, Y) = b(X, Y) = b(Y, X) = g(X, JY)$$

where b is the second fundamental form of M.

2) Let $R \in T_2^0(M)$ be the Ricci tensor of $(0,2)$-type and $ric \in T_1^1(M)$ defined by:

$$g(ricX, Y) = R(X, Y) \tag{4.3}$$

for every $X, Y \in \mathcal{X}(M)$. Then $J = ric$ is self-adjoint and:

Proposition 4.2. *If for every* $X, Y, Z \in \mathcal{X}(M)$:

$$(\nabla_X R)(Y, Z) = (\nabla_Y R)(X, Z) \tag{4.4}$$

then (4.2) *holds for* $J = ric$.

Proof. From (4.4):

$$\begin{aligned}&(\nabla_X R)(Y, Z) - (\nabla_Y R)(X, Z) = 0 = \\ &= X(R(Y, Z)) - Y(R(X, Z)) + R(X, \nabla_Y Z) - R(\nabla_X Z, Y) - R([X, Y], Z)\end{aligned}$$

and then:

$$\begin{aligned} R([X, Y], Z) = {} & X(R(Y, Z)) - Y(R(X, Z)) \\ & + R(X, \nabla_Y Z) - R(\nabla_X Z, Y). \end{aligned} \tag{4.5}$$

But:

$$(\nabla_X ric)Y - (\nabla_Y ric)X = \nabla_X ricY - \nabla_Y ricX - ric([X, Y])$$

and therefore:

$$\begin{aligned} & g((\nabla_X ric)Y - (\nabla_Y ric)X, Z) = \\ & = g(\nabla_X ricY, Z) - g(\nabla_Y ricX, Z) - g(ric[X, Y], Z) \\ & = X(g(ricY, Z)) - Y(g(ricX, Z)) - g(ricY, \nabla_X Z) \\ & + g(ricX, \nabla_Y Z) + R([X, Y], Z) \overset{(4.5)}{=} 0. \end{aligned}$$

Because $Z \in \mathcal{X}(M)$ is arbitrary we get the conclusion. □

5. Examples of symmetric non-metric linear connections

Necessary and sufficient conditions for a symmetric linear connection to be Levi-Civita with respect to a Riemannian metric are given in [16] and two examples of non-metric connections are presented in the cited paper. In this section we give other two examples using the differential system of autoparallel curves. It is notable that all the three papers cited in this section belong to physics oriented journals (Comm. Math. Phys., Rep. Math. Phys., J. Phys. A) and not to pure mathematical journals.

Example 5.1. (two-dimensional)
In (p. 100-101, [1]) it is proved that the differential system:

$$\begin{cases} \ddot{x} + \dfrac{y}{1+y^2}\dot{x}\dot{y} = 0 \\ \ddot{y} = 0 \end{cases} \tag{5.1}$$

may be naturally interpreted as the system of autoparallel curves for a torsion-free connection which is not a metric connection (two proofs are given). The first integrals of (5.1) are (see the arguments of next example):

$$F_1 = \dot{y}, \quad F_2 = \dot{x}\sqrt{1+y^2}. \tag{5.2}$$

Example 5.2. (three-dimensional)
In (p. 2188, [20]) it is proved that *the Halphen system*:

$$\begin{cases} \dot{x}_1 = x_2x_3 - x_3x_1 - x_1x_2 \\ \dot{x}_2 = x_3x_1 - x_1x_2 - x_2x_3 \\ \dot{x}_3 = x_1x_2 - x_2x_3 - x_3x_1 \end{cases} \tag{5.3}$$

admits no polynomial first integrals. Therefore, with the change $x_i \to \dot{x}_i$ we conclude that the differential system:

$$\begin{cases} \ddot{x}_1 = \dot{x}_2\dot{x}_3 - \dot{x}_3\dot{x}_1 - \dot{x}_1\dot{x}_2 \\ \ddot{x}_2 = \dot{x}_3\dot{x}_1 - \dot{x}_1\dot{x}_2 - \dot{x}_2\dot{x}_3 \\ \ddot{x}_3 = \dot{x}_1\dot{x}_2 - \dot{x}_2\dot{x}_3 - \dot{x}_3\dot{x}_1 \end{cases} \tag{5.4}$$

does not admit a polynomial first integral. If this symmetric connection will be Levi-Civita for the metric $g = (g_{ij})$ then the Hamiltonian $H = \frac{1}{2}g_{ij}\dot{x}_i\dot{x}_j$, which is 2-homogeneous, will be a first integral for (5.4), false.

Acknowledgments. I would like to thank Larry Bates and Alexey Tsygvintsev for sending their papers and to Vladimir Balan for useful remarks.

References

[1] Bates, L., Sniatycki, J.: *Nonholonomic reduction*, Rep. Math. Phys., 32 (1993), no. 1, 99-115.

[2] Balan, V.: *On deformation algebras*, Bull. Math. Soc. Sci. Math. Roumanie, 29 (77) (1985), no. 4, 291-196.

[3] Cruceanu, V., Fortuny, P., Gadea, P. M.: *A survey on paracomplex geometry*, Rocky Mountain J. Math., 26 (1996), no. 1, 83-115.

[4] Dragomir, S., Ornea, L.: *Locally Conformal Kähler Geometry*, Progress in Math., no. 155, Birkhäuser, Basel, 1998.

[5] Dubrovin, B.: *Geometry of 2D topological field theories*, in M. Francaviglia and S. Greco (Eds.), *Integrable Systems and Quantum Groups*, Springer LNM vol. 1620, 1996, 120-348.

[6] Dubrovin, B.: *Flat pencils of metrics and Frobenius manifolds*, math.DG/9803106.

[7] Gauduchon, P.: *Hermitian connections and Dirac operators*, Bol. U. M. I., ser. VII, vol. XI-B, supl. 2 (1997), 257-289.

[8] Golab, S.: *On semi-symmetric and quarter-symmetric linear connections*, Tensor, 29 (1975), 249-254.

[9] Herling, C.: *Multiplication on the tangent bundle*, math.DG/9910116.

[10] Lyra, G.: *Über eine modifikation der Riemannschen geometrie*, Math. Zeitschr., 54 (1951), 52-64.

[11] Manin, Y. I.: *Frobenius manifolds, Quantum Cohomology and Moduli Spaces*, AMS Colloquium, vol. 47, 1999.

[12] Miron, R., Atanasiu, Gh.: *Existence et arbitrariét é des connexions compatibles à une structure Riemann géné ralise du type presque k-horsymplectique métrique*, Kodai Math. J., 6 (1983), 228-237.

[13] Mishra, R. S., Pandey, S. N.: *On quarter-symmetric linear connections*, Tensor, 34 (1980), 1-7.

[14] Nicolescu, L.: *Quelques applications géomé triques des dérivations dans l'algèbre de déformation*, Tensor, 57 (1996), no. 1, 27-37.

[15] Peng, Lizhong, Yang, Lei: *The curl in seven dimensional space and its applications*, Approx. Theory & Its Appl., 15 (1999), no. 3, 66-80.

[16] Schmidt, B. G.: *Conditions on a connection to be a metric connection*, Comm. Math. Phys., 29 (1973), 55-59.

[17] Simon, U.: *On the inner geometry of the second fundamental form*, Michigan J. Math., 19 (1972), 129-132.

[18] Strachan, I.A.B.: *Frobenius submanifolds*, J. Geom. Phys., 38 (2001), no. 3-4, 285-307.

[19] Strachan, I. A. B.: *Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures*, math.DG/0201039.

[20] Tsygvintsev, A.: *On the existence of polynomial first integrals of quadratic homogeneous systems of ordinary differential equations* , J. Phys. A: Math. Gen., 34 (2001), 2185-2193.

[21] Vaisman, I.: *Sur quelques formules du calcul de Ricci global*, Comm. Math. Helvetici, 41 (1966-1967), 73-89.

[22] Vaisman, I.: *On locally conformal almost Kähler manifolds*, Israel J. Math., 24 (1976), 338-351.

ГРАНИЧНИ ФРОБЕНИУСОВИ СТРУКТУРИ ГЕНЕРАЦИИ СО ГЕОМЕТРИСКИ ДЕФОРМИРАНИ АЛГЕБРИ

Mircea Crâşmăreanu

Р е з и м е

Дадени се потребни и доволни услови за некои деформациони алгебри за сместување на формални Фробениусови структури. Исто така се презентирани примери за формални Фробениусови структури со фундаментален тензор кои не се деформационен тип и примери на симетрични и метрички сврзливости.

Faculty of Mathematics

University "Al. I. Cuza"

6600 Iaşi

Romania

e-mail: mcrasm@uaic.ro