Abstract. In [2] are considered $n$-Banach spaces, and in [4] are considered bounded and continuous linear $n$-functionals defined on $n$-normed space and several theorems connected with them, are proved. Then is proved that: Linear $n$-functional $F$ is continuous if and only if $F$ is bounded (theorem 4). In this paper, a dual space $X^*$ of space of bounded linear $n$-functionals is considered and it is proved that: if $X$ is $n$-Banach space than $(X^*, ||.||)$ is Banach space.

1. Introduction

Definition 1. Let $X_i, i = 1, 2, \ldots, n$ be linear subspace of same vector $n$-normed space. Then the mapping $F : X_1 \times \ldots \times X_n \rightarrow \mathbb{R}$ is called $n$-functional with domain $X_1 \times X_2 \times \ldots \times X_n$.

Definition 2. Let $F$ be $n$-functional with domain $X_1 \times X_2 \times \ldots \times X_n$. Then $F$ is linear $n$-functional if the following conditions are satisfied:

1. $F(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \sum_{z_i \in \{x_i, y_i\}} F(z_1, z_2, \ldots, z_n)$

2. $F(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_1 \alpha_2 \ldots \alpha_n F(x_1, x_2, \ldots, x_n)$

Definition 3. Let $X$ be $n$-normed space. Let $F$ be $n$-functional with domain $D(F) \subseteq X^n$ then $F$ is bounded if there exists real number $K \geq 0$ such that $F(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_1 \alpha_2 \ldots \alpha_n F(x_1, x_2, \ldots, x_n)$.

Let $F$ be bounded $n$-functional, we define norm of $F$, denoted by $||F||$, with

$$||F|| = \inf \{ K \mid ||F(x_1, x_2, \ldots, x_n)|| \leq K ||x_1, x_2, \ldots, x_n||, (x_1, x_2, \ldots, x_n) \in D(F) \}$$

(1)

If $F$ is unbounded $n$-functional, then we define $||F|| = +\infty$. 

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In this context for bounded linear n-functionals in [4] the following properties are proved.

**Lemma 1.** Let $F$ be a bounded linear $n$-functional and $x_i, i = 1, \ldots, n$, are linearly dependent vectors such that $(x_1, x_2, \ldots, x_n) \in D(F)$. Then $F(x_1, x_2, \ldots, x_n) = 0$.

**Theorem 1.** Let $F$ be a bounded linear $n$-functional on domain $D(F)$. Then

$$||F|| = \sup\{|F(x_1, x_2, \ldots, x_n)|; ||x_1, x_2, \ldots, x_n|| = 1, (x_1, x_2, \ldots, x_n) \in D(F)\}$$

Further on, continuity of linear $n$-functional is defined as following.

**Definition 4.** Let $F$ be $n$-functional. Then $F$ is continuous at the point $(x_1, x_2, \ldots, x_n)$ if for all $\varepsilon > 0$ exist $\delta > 0$ such that

$$|F(x_1, x_2, \ldots, x_n) - F(y_1, y_2, \ldots, y_n)| < \varepsilon$$

always when

$$||z_{1j}, z_{2j}, \ldots, z_{nj}|| < \delta$$

where

$$z_{ij} = \begin{cases} 
  x_i - y_i, & i = j \\
  x_i \vee y_i, & i \neq j 
\end{cases}$$

for $j = 1, 2, \ldots, n$. The $n$-functional $F$ is continuous if $F$ is continuous at every point from its domain.

In [4], for continuous $n$-functionals are proved the following properties.

**Theorem 2.** If the linear $n$-functional $F$ is continuous at the point $(0, 0, \ldots, 0)$, then $F$ is continuous at every point from its domain $D(F)$.

**Theorem 3.** Linear $n$-functional $F$ is continuous if and only if $F$ is bounded.

**Definition 5.** The sequence $\{x_k\}$ from the vector $n$-normed space $L$ is Cauchy sequence if there exists linear independent vectors $y_1, y_2, \ldots, y_n$ such that

$$\lim_{k, m \to \infty} ||x_k - x_m, y_2, \ldots, y_{n-1}, y_n|| = 0$$

$$\lim_{k, m \to \infty} ||x_k - x_m, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n|| = 0, \quad i = 2, \ldots, n-1$$

$$\lim_{k, m \to \infty} ||x_k - x_m, y_1, \ldots, y_{n-1}|| = 0.$$
Definition 6. The sequence \( \{x_k\} \) from \( n \)-normed space \( L \) is convergent if there exist \( x \in L \) such that
\[
\lim_{k \to \infty} \|x_k - x, y_1, \ldots, y_{n-1}\| = 0, \text{ for all } y_1, y_2, \ldots, y_{n-1} \in L.
\]

For \( x \) we shall say that is limit for the sequence \( \{x_k\} \) and we’ll write \( x_k \to x, k \to \infty \).

Definition 7. For \( n \)-normed space \( L \), well say that is \( n \)-Banach space if every Cauchy sequence is convergent.

Theorem 4. Every real \( n \)-normed vector space with dimension \( n \) is \( n \)-Banach space.

2. DUAL SPACE OF THE SPACE OF BOUNDED LINEAR \( n \)-FUNCTIONALS

Definition 8. Let \( X \) be \( n \)-Banach space, \( X^* \) is a set of bounded linear \( n \)-functionals on domain \( X^n \) and let \( F, G \in X^* \). We define

a) \( F = G \) if \( F(x_1, x_2, \ldots, x_n) = G(x_1, x_2, \ldots, x_n) \), for all \( (x_1, x_2, \ldots, x_n) \in X^n \),

b) \( (F + G)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n) \), for all \( (x_1, x_2, \ldots, x_n) \in X^n \),
c) \( (\alpha F)(x_1, x_2, \ldots, x_n) = \alpha F(x_1, x_2, \ldots, x_n) \), for all \( \alpha \) and all \( (x_1, x_2, \ldots, x_n) \in X^n \).

Theorem 5. Let \( X \) be \( n \)-Banach space. Then \( (X^*, ||.||) \) is Banach space.

Proof. Let \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n \) and \( \alpha_i \in \mathbb{R}, i = 1, 2, \ldots, n \). Then according to Definition 2. we have
\[
(F + G)(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = F(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) + G(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) =
= \sum_{z_i \in \{x_i, y_i\}} F(z_1, z_2, \ldots, z_n) + \sum_{z_i \in \{x_i, y_i\}} G(z_1, z_2, \ldots, z_n) =
= \sum_{z_i \in \{x_i, y_i\}} (F + G)(z_1, z_2, \ldots, z_n)
\]
\[
(F + G)(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) =
= F(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) + G(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) =
= \alpha_1 \alpha_2 \ldots \alpha_n F(x_1, x_2, \ldots, x_n) + \alpha_1 \alpha_2 \ldots \alpha_n G(x_1, x_2, \ldots, x_n) =
= \alpha_1 \alpha_2 \ldots \alpha_n [F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)] =
= \alpha_1 \alpha_2 \ldots \alpha_n (F + G)(x_1, x_2, \ldots, x_n).
\]
Further on, because of Definition 3 we have
\[
\| (F + G)(x_1, x_2, \ldots, x_n) \| = |F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)| \\
\leq |F(x_1, x_2, \ldots, x_n)| + |G(x_1, x_2, \ldots, x_n)| \\
\leq \| F \| \cdot \| x_1, x_2, \ldots, x_n \| + \| G \| \cdot \| x_1, x_2, \ldots, x_n \| \\
= (\| F \| + \| G \|) \cdot \| x_1, x_2, \ldots, x_n \|
\]
which means that \( F + G \in X^* \) and clearly \( \| F + G \| \leq \| F \| + \| G \| \).

Analogously we can prove that for every \( F \in X^* \), \( \alpha F \in X^* \) and \( \| \alpha F \| = |\alpha| \cdot \| F \| \) holds.

From the other hand, according to Definition 3 we have
\[
|F(x_1, x_2, \ldots, x_n)| \leq \| F \| \cdot \| x_1, x_2, \ldots, x_n \|, \quad \text{for all } (x_1, x_2, \ldots, x_n) \in X^n,
\]
so \( \| F \| = 0 \) if and only if \( F = 0 \), which means that \( X^* \) is vector space with norm defined by (1).

Let \( \{ F_k \} \) be Cauchy sequence on \( X^* \), i.e. let
\[
\lim_{m \to \infty} \| F_k - F_m \| = 0 \quad (2)
\]

Then for all \( (x_1, x_2, \ldots, x_n) \in X^n \) is true that
\[
\| F_k(x_1, x_2, \ldots, x_n) - F_m(x_1, x_2, \ldots, x_n) \| \leq \| F_k - F_m \| \cdot \| x_1, x_2, \ldots, x_n \|
\]
which means that for every \( (x_1, x_2, \ldots, x_n) \in X^n \) the real sequence
\( \{ F_k(x_1, x_2, \ldots, x_n) \} \) is a Cauchy sequence. On \( X^n \) let define functional \( F \) with
\[
F(x_1, x_2, \ldots, x_n) = \lim_{k \to \infty} F_k(x_1, x_2, \ldots, x_n), \quad (x_1, x_2, \ldots, x_n) \in X^n.
\]

Then, for all \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n \) and \( \alpha_i \in \mathbb{R}, i = 1, 2, \ldots, n \) we have
\[
F(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \lim_{k \to \infty} F_k(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)
\]
\[
= \lim_{k \to \infty} \sum_{z_i \in \{x_i, y_i\}} F_k(z_1, z_2, \ldots, z_n)
\]
\[
= \sum_{z_i \in \{x_i, y_i\}} \lim_{k \to \infty} F_k(z_1, z_2, \ldots, z_n)
\]
\[
= \sum_{z_i \in \{x_i, y_i\}} F(z_1, z_2, \ldots, z_n)
\]
and
\[
F(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \lim_{k \to \infty} F_k(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)
\]
\[
= \lim_{k \to \infty} \alpha_1 \alpha_2 \ldots \alpha_n F_k(x_1, x_2, \ldots, x_n)
\]
\[
= \alpha_1 \alpha_2 \ldots \alpha_n \lim_{k \to \infty} F(x_1, x_2, \ldots, x_n)
\]
\[
= \alpha_1 \alpha_2 \ldots \alpha_n F(x_1, x_2, \ldots, x_n).
\]
i.e. \( F \) is \( n \)-linear functional. On the other hand, for the sequence \( \{F_k\} \), \(| |F_k|| - ||F_m||| \leq ||F_k - F_m|| \) holds.

Now from (2) we get that \( \{||F_k||\} \) is real Cauchy sequence, which means that there exist \( K \in \mathbb{R} \) such that \( ||F_k|| \leq K \), for all \( k \in \mathbb{N} \), from where we get

\[
|F(x_1, x_2, \ldots, x_n)| = \limsup_{k \to \infty} F_k(x_1, x_2, \ldots, x_n)
\]

\[
= \limsup_{k \to \infty} |F_k(x_1, x_2, \ldots, x_n)|
\]

\[
\leq \limsup_{k \to \infty} ||F_k|| \cdot ||x_1, x_2, \ldots, x_n||
\]

\[
\leq K||x_1, x_2, \ldots, x_n||,
\]

i.e. \( F \in X^* \).

We’ll prove that \( \{F_k\} \) converges to \( F \). Let \( ||x_1, x_2, \ldots, x_n|| \neq 0 \). If \( \varepsilon > 0 \) is chosen, then from (2) we have that there exist \( n_0 \in \mathbb{N} \) such that \( ||F_m - F_k|| < \varepsilon \) when \( m, k > n_0 \), so by Definition 3 we have

\[
|F_m(x_1, x_2, \ldots, x_n) - F_k(x_1, x_2, \ldots, x_n)| \leq ||F_m - F_k|| \cdot ||x_1, x_2, \ldots, x_n||
\]

\[
\leq \varepsilon ||x_1, x_2, \ldots, x_n||,
\]

for all \( m, k \geq n_0 \). On the other hand, because of

\[
F(x_1, x_2, \ldots, x_n) = \lim_{k \to \infty} F_k(x_1, x_2, \ldots, x_n)
\]

there exist \( M = M(x_1, x_2, \ldots, x_n) > n_0 \) such that

\[
|F_M(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| < \varepsilon ||x_1, x_2, \ldots, x_n||.
\]

So we have

\[
|F_k(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \leq
\]

\[
\leq |F_k(x_1, x_2, \ldots, x_n) - F_M(x_1, x_2, \ldots, x_n)| +
\]

\[
+ |F_M(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)|
\]

\[
\leq \varepsilon ||x_1, x_2, \ldots, x_n|| + \varepsilon ||x_1, x_2, \ldots, x_n|| = 2 \cdot \varepsilon ||x_1, x_2, \ldots, x_n||
\]

for \( k > n_0 \). If \( ||x_1, x_2, \ldots, x_n|| = 0 \), then the vectors \( x_1, x_2, \ldots, x_n \) are linearly dependent, and according to Lemma 1 it follows that

\[
F_k(x_1, x_2, \ldots, x_n) = 0 = F(x_1, x_2, \ldots, x_n)
\]

which means \( |F_k(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \leq 2 \cdot \varepsilon ||x_1, x_2, \ldots, x_n|| \), for all \( k > n_0 \). Hence, for all \( (x_1, x_2, \ldots, x_n) \in X^n \) the following holds

\[
|F_k(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)| \leq 2 \cdot \varepsilon ||x_1, x_2, \ldots, x_n||,
\]

for all \( k > n_0 \). i.e. accordingly to Definition 3 we get \( ||F_k - F|| \leq 2\varepsilon \), for \( k > n_0 \), i.e. \( \{F_k\} \) converge to \( F \).

Finally from the arbitrariness of the Cauchy sequence \( \{F_k\} \) we have that \( (X^*, ||.||) \) is Banach space. 

\( \square \)
REFERENCES
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