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FOR THE ZEROS OF THE SOLUTIONS OF THE ELEMENTARY VEKUA EQUATIONS

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Abstract. After the year of 2000 it is noticed a huge trend in the intensity of the papers which are dealing with valuation of the zeros of the complex differential equations, especially the equation of the "complex oscillations" $\frac{d^2W}{dz^2} + A(z)W = 0$, [1], [2].

For the Vekua type equations

$$\frac{\hat{d}W}{d\bar{z}} = A(z,\bar{z})W + B(z,\bar{z})\overline{W} + F(z,\bar{z})$$

which are similar to them, but in the space of two complex variables $z, \bar{z}$ we have not noticed such trend in the study of the zeros of the solutions. In this paper we are giving theorems for existence of the zeros of the special Vekua equation:

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1. INTRODUCTION

In 1909, while he was solving a problem from the theory of elasticity, G.V.Kolosov [4] introduced the expressions

$\frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{dz}$ and $\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{d\bar{z}}$
known as operator derivatives of the complex function $W = W(z) = u(x,y) + +iv(x,y)$ from the complex variable $z = x + iy$ and $\bar{z} = x - iy$, corresponding. The operating rules for these derivatives are completely given in the monograph of Г. Н. Положий [5] (page18-31). In the mentioned monograph are also defined so cold operator integrals

$$\overset{\wedge}{\int} f(z)dz \text{ and } \overset{\wedge}{\int} f(z)d\bar{z}$$

from $z = x+iy$ and $\bar{z} = x-iy$, corresponding, from the complex function $f = f(z)$ in the area $D \subseteq \mathbb{C}$. Here their operating rules are proved (page 32 - 41).

Let $u = u(x,y)$ and $v = v(x,y)$ are two real differentiable functions from real variables x and y and let $x + iy = z$. Usually the complex function $W = W(z)$

is introduced as a linear combination of the functions u and v, i.e. $W = W(z) = u(x, y) + iv(x, y)$.

Using the formulas $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we have

$$W = u(x, y) + iv(x, y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = W(z, \bar{z})$$

It means that the function W generally doesn't depend only from z but from $\bar{z}$, as well, so it seems that more correct is to write $W = W(z, \bar{z}) = u(x, y) + +iv(x, y)$. For the functions $z = x + iy$ and $\bar{z} = x - iy$ we have the condition

$$\begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial \bar{z}}{\partial x} & \frac{\partial \bar{z}}{\partial y} \end{vmatrix} = -2i \neq 0$$

which means that they are independent one from the other. This expression of the function is completely justified, [6], so we will use it in this paper.

It can be shown that $\frac{\hat{d}W}{dz} = \frac{\partial W}{\partial z}$ and $\frac{\hat{d}W}{d\bar{z}} = \frac{\partial W}{\partial \bar{z}}$, i.e. the operator derivative coincides with the partial derivative of the function W by the variables z and $\bar{z}$, corresponding. As for the operator integrals, here we have to mention that for example for the operator integral by z, the variable $\bar{z}$ is acting as a constant and vice versa.

2. Quadratures I.N.Vekua.

The simplest Vekua equation

$$\frac{\partial W}{\partial \bar{z}} = F(z, \bar{z}), \tag{1}$$

where F is a continuous differentiable function from two independent complex variables z and $\bar{z}$, is solved as it is known ([7], [8]) through generalized quadratures:

$$W(z, \bar{z}) = \hat{\int} F(z, \bar{z}) d\bar{z} + C(z) \tag{2}$$

where:

1^0 $\hat{\int} F(z, \bar{z})\, d\bar{z}$ means that it is integrated by $\bar{z}$, while z is considered as a constant.

2^0 Here the integration "constant" $C(z)$ is not a common complex number, but it is a "generalized constant" - an arbitrary analytic function from z, $C(z) = \alpha(x, y) + i\beta(x, y)$, where $\alpha'_x = \beta'_y$ and $\alpha'_y = -\beta'_x$.

If we put

$$F(z, \bar{z}) = f(x, y) + ig(x, y), \quad d\bar{z} = dx - idy$$

then for the solution (2) we have

$$W(z, \bar{z}) = \hat{\int} (f + ig)(dx - idy) + \alpha + i\beta = \hat{\int} [(fdx + gdy) + i(gdx - fdy)] + \alpha + i\beta$$

and for its zeros we have the following system:

$$\int_L [f(x,y)dx + g(x,y)dy] = -\alpha(x,y) \text{ and } \int_L [g(x,y)dx - f(x,y)dy] = -\beta(x,y).$$

Here L is an arbitrary smooth line which bonds two spots $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$ in the considered area D. Since f and g are not parts of an analytic function, that's why this integrals depend of L, but they always determine some continuous functions Φ and Ψ:

$$\Phi\left(x,y\right) = -\alpha\left(x,y\right), \quad \Psi\left(x,y\right) = -\beta\left(x,y\right).$$

If we solve the new equations explicitly by y, according to the theorem for existence of an implicit function, then with small restrictions we get

$$L_1 : y = \varphi\left(x\right), \quad L_2 : y = \chi\left(x\right). \tag{3}$$

The first line L_1 is continuous and smooth line by which $u = \mathrm{Re}W = 0$ and there for it is called a line of the **u-zeros**. The second line L_2 is similar to the first, but for $u = \mathrm{Re}W = 0$ and it is called a line of the **v-zeros**.

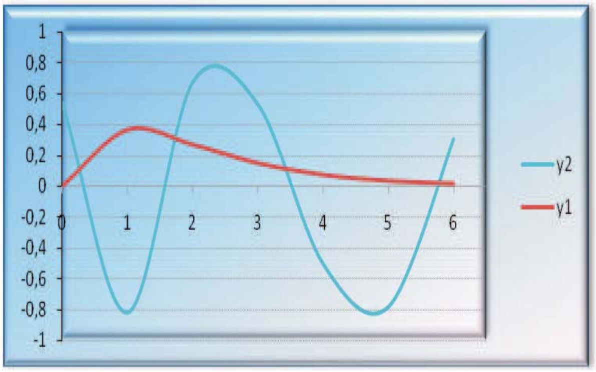
The zeros of $W = u + iv$ shall belong to the intersection of the lines L_1 and L_2. Since the intersection of two smooth lines is some row of spots, we have:

Theorem 1: The zeros of the solution of the Vekua equation (1) are in the intersection $L_1 \cap L_2$ of the smooth curves L_1 and L_2, and their number and characteristics are depending from the continuous function $F(z, \bar{z})$ and from the analytic function $C(z)$.

Example 1. The function $W = \bar{z} + e^z$ is a particular solution of the Vekua equation $\dfrac{\partial W}{\partial \bar{z}} = 1$. According to the procedure mentioned before we have the system

$$u = e^x \cos y + x = 0$$
$$v = e^x \sin y - y = 0$$

Eliminating y we get transcendent equation by x
$\dfrac{x}{e^x} = \cos\sqrt{e^{2x} - x^2}$ for the abscissas of the zeros of the solution $W = \bar{z} + e^z$ of the Vekua equation $\dfrac{\partial W}{\partial \bar{z}} = 1$. If we solve the transcendent equation graphically, we get typical Sturm zeros of the oscillatory equation $y_1 = y_2$, where $y_1 = xe^{-x}, y_2 = \cos\sqrt{e^{2x} - x^2}$. Finally, the considered solution of the equation in this example has countless zeros and all of them are isolated.



Example 2. The function $W = \bar{z} - z^2$ is a particular solution of the same equation from example 1. From $W = 0$ we get system for the zeros

$$\begin{aligned} & x - x^2 + y^2 = 0 \\ & 2xy + y = 0 \end{aligned}$$

whose solution are the four isolated zeros $M_1\left(0,0\right), M_2\left(1,0\right)$, $M_3(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $M_4(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. We see that:

I. Finding the zeros of the solutions of an areolar equation is not a trivial task.

II. The zeros depend from the coefficients, which is natural, but from the integration constant $C(z)$, too and it is an essential dependence.

III. The zeros may be isolated or not. For example, one of the solutions of $\frac{\partial W}{\partial \bar{z}} = z$ is $W = |z|^2 - 1$ and the zeros are on the circle $|z| = 1$.

Areolar linear homogenous differential equation from I order. It is the equation

$$\frac{\partial W}{\partial \bar{z}} = A\left(z, \bar{z}\right) W \tag{4}$$

It is known that it can be solved with quadratures:

$$\frac{\partial W/\partial \bar{z}}{W} = A\left(z, \bar{z}\right),$$

$$\hat{\int} \frac{\partial}{\partial \bar{z}} \left(\ln W\left(z, \bar{z}\right)\right) d\bar{z} = \hat{\int} A\left(z, \bar{z}\right) d\bar{z}$$

where from we get

$$W\left(z, \bar{z}\right) = C\left(z\right) e^{\hat{\int} A(z,\bar{z}) d\bar{z}} \tag{5}$$

Here $C = C(z)$ is an arbitrary analytic function in the role of an integrating constant and with $\hat{\int} A(z,\bar{z})\,d\bar{z}$ we denote one operatory primitive function from $\bar{z}$ of the function $A = A(z,\bar{z})$.
Next we have an important

Theorem 2: The zeros of the solutions of the equation (4) (i.e. the common zeros of $u(x,y) = \mathrm{Re}W = 0$ and $v(x,y) = \mathrm{Im}W = 0$) are in:

1^0. The zeros of the analytic function $C(z)$, whose determination will depend from given boundary conditions.

2^0. Eventually in the zeros of the second multiplier

$$e^{\hat{\int} A(z,\bar{z})d\bar{z}} \tag{6}$$

Lets now explore the second part 2^0. It is essential to separate the real from imaginary part:

$$I = \hat{\int} A(z,\bar{z})\,d\bar{z} = \int_\Gamma (a+ib)(dx - idy) = \int_\Gamma ([adx + bdy] + i[bdx - ady]).$$

For (6) we have:

$$\begin{aligned} e^{\hat{\int} A(z,\bar{z})d\bar{z}} &= e^{\int_\Gamma [adx+bdy]} e^{\int_\Gamma i[bdx-ady]} = \\ &= e^{\int_\Gamma [adx+bdy]}\left[\cos\int_\Gamma [bdx - ady] + i\sin\int_\Gamma [bdx - ady]\right] = 0. \end{aligned}$$

Since $\exp\int_\Gamma [adx + bdy] > 0$, we have that the zeros of (6) can be looked in the system

$$\begin{cases} \cos\int_\Gamma [bdx - ady] = 0 \\ \sin\int_\Gamma [bdx - ady] = 0 \end{cases} \tag{7}$$

From (7) we get

$$bdx - ady = (2k-1)\,{}^{\pi}\!/_{2}, \quad k = 1,2,3,\ldots \tag{8}$$

$$bdx - ady = n\pi, \quad n = 0,1,2,3,\ldots \tag{9}$$

Since the left sides in this system are equal, and the right sides are either odd number multiplied with ${}^{\pi}\!/_{2}$, or whole number by π - they are always different. So we have that (7) doesn‘t have solutions for any a and b.

Theorem 3: The function $A = A(z,\bar{z})$ from two complex variables doesn‘t give zeros in the solution (5) of the basic Vekua equation (4).

Example 3. Lets explore the Vekua equation $\dfrac{\partial W}{\partial \bar{z}} = z^2 W$. According to the Theorem 3 the function $A = A(z, \bar{z}) = z^2$ doesn‘t give zeros in the solution

$$\begin{gathered} W(z, \bar{z}) = C(z)\, e^{\hat{\int} z^2 d\bar{z}} = C(z)\, e^{z(z\bar{z})} = C(z)\, e^{z|z|^2} = C(z)\, e^{x\left(x^2+y^2\right)} \cdot e^{iy\left(x^2+y^2\right)} = \\ = e^{x\left(x^2+y^2\right)} C(z) \left[\cos y\left(x^2+y^2\right) + i \sin y\left(x^2+y^2\right)\right] \end{gathered}$$

of the considered equation. We get the following conclusions:

1^0. The zeros of the solution $W = W(z, \bar{z})$ are in the zeros of the analytic function $C = C(z)$ and they are isolated.

2^0. For the u-zeros of the solution $W = W(z, \bar{z})$ we have $\cos y\left(x^2+y^2\right) = 0$, where from we get a family of cubic curves

$$y^3 + yx^2 = (2k-1)\, {}^{\pi}\!/_{2}, \quad k = 1, 2, 3, \ldots$$

and they are not isolated; Similar, for the v-zeros of the solution we have $\sin y\left(x^2+y^2\right) = 0$ i.e. not isolated zeros determined with the equation

$$y^3 + yx^2 = n\pi, \quad n = 0, 1, 2, 3, \ldots.$$

Zeros of the solutions of the no homogenous linear Vekua equation from I order. Let be given the Vekua equation in the next shape

$$\frac{\partial W}{\partial \bar{z}} = A(z, \bar{z})\, W + F(z, \bar{z}) \tag{10}$$

(so to be precise without $\overline{W}$). We can see that the zeros shall depend from three sources: an analytic function $C(z)$ and continuous not analytic coefficients $A(z, \bar{z})$ and $F(z, \bar{z})$.

It is known that ([5], page 39-40) the equation (10) has the following general solution

$$W(z, \bar{z}) = e^{\hat{\int} A(z,\bar{z}) d\bar{z}} \left[C(z) + \hat{\int} F(z, \bar{z})\, e^{-\hat{\int} A(z,\bar{z}) d\bar{z}} d\bar{z} \right]. \tag{11}$$

As for the zeros, we already know that the multiplier $e^{\hat{\int} A d\bar{z}}$ doesn‘t have zeros, the only zeros are those of its real or its imaginary part. More valuable in the sense of the zeros is the second multiplier in (11), where figures the general analytic function $C(z)$. That‘s why in (11) we should consider the both aspects together.

If we mark

$$\left\{ \begin{array}{l} A(z,\bar{z}) = a(x,y) + ib(x,y); \quad a, b \ - \ \text{continous} \\ F(z,\bar{z}) = f(x,y) + ig(x,y); \quad f, g \ - \ \text{continous} \\ C(z) = \alpha(x,y) + i\beta(x,y) - \text{analytic, with the} \\ \ \text{conditions} \ \alpha'_x = \beta'_y, \alpha'_y = -\beta'_x \end{array} \right. \tag{12}$$

then with a little technique, separating the real and the imaginary part from (11), we get the conditions for u-zeros and for the v-zeros:

$$\begin{aligned}
&\cos\int_\Gamma (bdx-ady)\cdot\int_\Gamma\left\{e^{-\int_\Gamma(adx+bdy)}\left[(fdx+gdy)\cos\int_\Gamma(bdx-ady)-\right.\right.\\
&-\left.\left.(gdx-fdy)\sin\int_\Gamma(bdx-ady)\right]\right\}-\\
&-\sin\int_\Gamma (bdx-ady)\cdot\int_\Gamma\left\{e^{-\int_\Gamma(adx+bdy)}\left[(gdx-fdy)\cos\int_\Gamma(bdx-ady)+\right.\right.\\
&+\left.\left.(fdx+gdy)\sin\int_\Gamma(bdx-ady)\right]\right\}=\\
&=\beta(x,y)\sin\int_\Gamma(bdx-ady)-\alpha(x,y)\cos\int_\Gamma(bdx-ady)
\end{aligned}\tag{13}$$

and

$$\begin{aligned}
&\sin\int_\Gamma (bdx-ady)\cdot\int_\Gamma\left\{e^{-\int_\Gamma(adx+bdy)}\left[(fdx+gdy)\cos\int_\Gamma(bdx-ady)-\right.\right.\\
&-\left.\left.(gdx-fdy)\sin\int_\Gamma(bdx-ady)\right]\right\}+\\
&+\cos\int_\Gamma (bdx-ady)\cdot\int_\Gamma\left\{e^{-\int_\Gamma(adx+bdy)}\left[(gdx-fdy)\cos\int_\Gamma(bdx-ady)+\right.\right.\\
&+\left.\left.(fdx+gdy)\sin\int_\Gamma(bdx-ady)\right]\right\}=\\
&=-\alpha(x,y)\sin\int_\Gamma(bdx-ady)-\beta(x,y)\cos\int_\Gamma(bdx-ady)
\end{aligned}\tag{14}$$

This equations (13) and (14) represent continuous bonds between six different real functions given with (12) from two real independent variables (x, y). So (13) and (14) are plane continuous curves:

$$\Gamma_1\,(x,y;a,b,f,g,\alpha,\beta)=0 \tag{13.1}$$

$$\Gamma_2\,(x,y;a,b,f,g,\alpha,\beta)=0 \tag{14.1}$$

and each of them is a geometric place of the zeros: (13.1) for the zeros of $u = \mathrm{Re}W$ and (14.1) for the zeros of $v = \mathrm{Im}W$. We have

Theorem 4: The equations for the zeros of the real and imaginary part of the solution (11) of the equation (10), are given with (13.1) and (14.1), corresponding. Here this zeros are not isolated.

Common zeros

The curves (13.1) and (14.1) generally are different and they may intersect, but it is not necessary. According to this the function $W(z,\bar z) = u + iv$, determined with (11), will have zeros, if $u(x, y) = 0$ and $v(x, y) = 0$ at the same time.

Geometrically, this means that there exists intersection of the curves

$$\Gamma_1\,(x,y;a,b,f,g,\alpha,\beta)=0\cap\Gamma_2\,(x,y;a,b,f,g,\alpha,\beta)=0 \tag{15}$$

or shortly $\Gamma_1 = 0 \cap \Gamma_2 = 0$. These are spots (x_i, y_i), which will depend from $(a, b; f, g; \alpha, \beta)$ and the operations in (13) and (14).

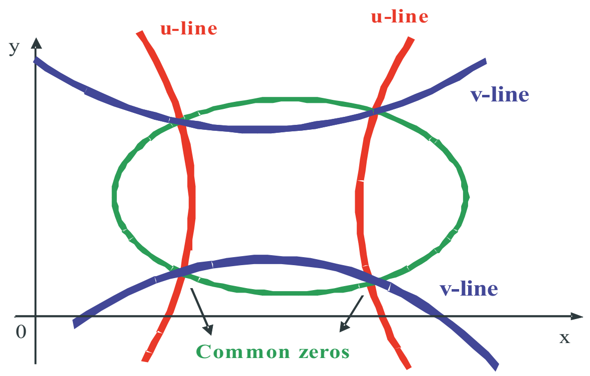
So, let (13) and (14) are fulfilled for some choice of $(a, b; f, g; \alpha, \beta) : \Gamma_1 = 0$ and $\Gamma_2 = 0$. We are asking which are the common zeros of $u(x, y) = \mathrm{Re}W = 0$ and of $v(x, y) = \mathrm{Im}W = 0$. In mathematics there isn't unanimous response to this and similar questions, because it depends of many functions and conditions for them. That is why we form various sufficient conditions for the common zeros, for example the functions $u + v, u \cdot v, \sqrt{u^2 + v^2}, \lambda u + \beta v, \ldots$, which equalled to zero, are satisfied separately for $u = 0$ and for $v = 0$; but at the same time they contain much more spots, which aren't zeros.

From the intersection (15) we have, if we form new function

$$\Gamma_1 + \Gamma_2 = 0 \tag{16}$$

as a sum of two continuous curves, it is continuous and satisfied from the zeros of $\Gamma_1 = 0$ and from the zeros of $\Gamma_2 = 0$, but here in (16) there are much more spots than there are common zeros of Γ_1 or of Γ_2.

The geometric interpretation is given in the picture below.



Theorem 5: One of the many sufficient conditions for the common zeros is fulfilling the condition which is a sum of the formulas (13) and (14) for the zeros only of $u(x, y)$ and only of $v(x, y)$.

Since this condition depends of many other conditions, we won't write about it generally, but we will consider some special cases.

Special cases

I. $A(z, \bar{z}) = 0$

If $A = 0$, i.e. $a = 0$ and $b = 0$, then the conditions (13) and (14) are simplified in great deal. We get:

I. for the zeros of $u(x, y) = 0$ a relation

$$\int_{\Gamma} (f dx + g dy) = -\alpha(x, y) \tag{17}$$

II. for the zeros of $v(x, y) = 0$ a relation

$$\int_{\Gamma} (g dx - f dy) = -\beta(x, y) \tag{18}$$

and according to (16), sufficient condition for the common zeros

$$\text{III} \quad \int_{\Gamma} [(f + g)\, dx + (g - f)\, dy] = -(\alpha + \beta) \tag{19}$$

From (19), using the Cauchy-Riemann conditions for $C(z) = \alpha + i\beta$; $\alpha'_x = \beta'_y$, $\beta'_x = -\alpha'_y$ and the theorem of Laplace for finding the differential under the sign of line integral, we get only a special theorem:

Theorem 6: In order the special Vekua equation $\dfrac{\partial W}{\partial \bar{z}} = F(z, \bar{z})$ to have a solution W which has zeros, it is sufficient

1) the function $F(z, \bar{z})$ to be real, i.e. $g(x, y) = 0$,

2) and $f(x, y) = \alpha'_y - \alpha'_x$

From this kind of theorems with small generality (and highly specialized), we can‘t be satisfied. That is why we don‘t use the intersection (15) in analytic form (13)+(14), but we will work only the separate zeros; i.e. only with (17) and (18). If we use again the Laplasians $\triangle\alpha = 0$ and $\triangle\beta = 0$, and the theorem for finding the differential under the sign of the integral, we have much more general:

Theorem 7: In order the solution $W(z, \bar{z})$ of the Vekua equation $\dfrac{\partial W}{\partial \bar{z}} = F(z, \bar{z}) = f + ig$ to have zeros it is sufficient f and g to be harmonic functions.

II. $F(z, \bar{z}) = 0$

In this case the Vekua equation (10) is reduced to the areolar homogeneous differential linear equation (4). For this equation we have the Theorem 3.

III. Relations between a and b which make possible solving the equation with quadratures

From the relations (13) and (14) which contain the expression $e^{-\int (adx+bdy)}$ it is obvious that we will work much more simplified if we have

$$a(x, y)dx + b(x, y)dy = 0 \tag{20}$$

If this equation has quadrature solution in the next shape:

$$y = \Phi(x, C) \tag{21}$$

which will be the case if (20) is a differential equation in total differential i.e. if it's valid the condition:

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x} \tag{22}$$

then the solution of (20) is implicitly determined with:

$$\int a(x,y)dx + \int \left[b(x,y) - \frac{\partial}{\partial y}\int a(x,y)dx\right] dy = Const. = C \tag{23}$$

and we have

Theorem 8: If the equation (20) is an equation with total differential, i.e. if it is valid (22), and if from (23) the implicitly determined function $y = y(x)$ is written in explicit shape (21), then from (13) and (14) is possible explicit separation of common u-zeros and v-zeros of the solution of the Vekua equation (10).

IV. Zeros only of u and only of v

From (13) and (14) we can see that the integral

$$\varphi = \int_{\Gamma} (bdx - ady) \tag{24}$$

has a big influence over the zeros and it is the argument of the function $\sin\varphi$ and $\cos\varphi$. If we put

$$b(x,y)dx - a(x,y)dy = 0 \tag{25}$$

then $\cos\varphi = 1$, $\sin\varphi = 0$ and the conditions for the zeros of $u(x,y) = 0$ and $v(x,y) = 0$ are

$$\int_{\Gamma} e^{-\int_{\Gamma}(adx+bdy)} (fdx + gdy) = -\alpha(x,y) \tag{26}$$

$$\int_{\Gamma} e^{-\int_{\Gamma}(adx+bdy)} (gdx - fdy) = -\beta(x,y) \tag{27}$$

If the equation (25) is solvable with quadratures, i.e. its general solution is

$$y = \psi(x, C) \tag{28}$$

than with the help of this solution we will form the equation for location of the common zeros. Here

$$adx + bdy = adx + b\frac{b}{a}dx = \frac{a^2+b^2}{a}dx \tag{29}$$

and from (26) and (27) we get

$$\int_{\Gamma} e^{-\int \frac{a^2+b^2}{a}dx} [fdx + gdy] = -\alpha(x,y) \tag{30}$$

$$\int\limits_{\Gamma} e^{-\int \frac{a^2+b^2}{a}dx}\left[gdx - fdy\right] = -\beta(x,y) \tag{31}$$

If we put total differential in (30) and (31) and if we equal the expressions in front of dx and dy, corresponding, we get

$$\begin{cases} \exp\left(-\int \frac{a^2+b^2}{a}dx\right)\cdot f = -\alpha'_x \\ \exp\left(-\int \frac{a^2+b^2}{a}dx\right)\cdot g = -\alpha'_y \end{cases} \tag{32}$$

and

$$\begin{cases} \exp\left(-\int \frac{a^2+b^2}{a}dx\right)\cdot g = -\beta'_x \\ \exp\left(-\int \frac{a^2+b^2}{a}dx\right)\cdot (-f) = -\beta'_y \end{cases} \tag{33}$$

From the both systems, if we divide the equations we get quotients which according to the Cauchy-Riemann conditions for α and β, are equal to $\dfrac{f}{g}$. For f and g which depend from $a = a(x,y)$ and $b = b(x,y)$ we have:

$$f(x,y) = \beta'_y e^{\int \frac{a^2+b^2}{a}dx}$$

$$g(x,y) = \beta'_x e^{\int \frac{a^2+b^2}{a}dx}$$

In this case we get equations for abscissas of the zeros specially for $u = \mathrm{Re}W$ and for $v = \mathrm{Im}W$

$$f\left(x,\psi\left(x\right)\right) - \beta'_y\left(x,\psi\right)\exp\int\frac{a^2\left(x,\psi\left(x\right)\right)+b^2\left(x,\psi\left(x\right)\right)}{a\left(x,\psi\right)}dx = 0 \tag{34}$$

and

$$g\left(x,\psi\left(x\right)\right) - \beta'_x\left(x,\psi\right)\exp\int\frac{a^2\left(x,\psi\left(x\right)\right)+b^2\left(x,\psi\left(x\right)\right)}{a\left(x,\psi\right)}dx = 0 \tag{35}$$

Theorem 9: In order the solution (11) of the Vekua equation (10) to have separate u-zeros and separate v-zeros of the solution $W = u + iv$, and with that the possibility for common zeros, under the condition that the coefficient $A(z,\bar{z}) = a + ib$ is a solution of the differential equation (25), it is necessary that the second coefficient $F(z,\bar{z}) = f + ig$ to be determined from the formulas (32) and (33). Here $\alpha + i\beta = C(z)$ is an arbitrary analytic function. The locations of the zeros are given with the equations (34) and (35).

V. Appearance of harmonic oscilations

The expressions $\cos\int\limits_{\Gamma}(bdx - ady)$ and $\sin\int\limits_{\Gamma}(bdx - ady)$ in the conditions (13) and (14) for separate zeros define complex oscillatory functions from two independent variables (x,y) . From theoretic and useful reasons it is important the case when they determine simple harmonic oscillations. Obviously, for that we must have

$$bdx - ady = \lambda dx, \quad \lambda = const \tag{36}$$

because then

$$\int (bdx - ady) = \int \lambda dx = \lambda x$$

and so we would have harmonic oscillations $\cos \lambda x$, $\sin \lambda x$ with a period $T = \frac{2\pi}{\lambda}$. But, (36) is one differential equation:

$$\frac{dy}{dx} = \frac{b(x,y) - \lambda}{a(x,y)} \tag{37}$$

(for $\lambda = 0$ we have the previous case IV.) The conditions (13) and (14) are

$$\begin{aligned}&\cos \lambda x \cdot \int_\Gamma \left\{ e^{-\int \frac{a^2+b^2}{a} dx} \left[(fdx + gdy) \cos \lambda x - (gdx - fdy) \sin \lambda x \right] \right\} - \\ &- \sin \lambda x \cdot \int_\Gamma \left\{ e^{-\int \frac{a^2+b^2}{a} dx} \left[(gdx - fdy) \cos \lambda x + (fdx + gdy) \sin \lambda x \right] \right\} = \\ &= \beta(x,y) \sin \lambda x - \alpha(x,y) \cos \lambda x\end{aligned} \tag{38}$$

and

$$\begin{aligned}&\sin \lambda x \cdot \int_\Gamma \left\{ e^{-\int \frac{a^2+b^2}{a} dx} \left[(fdx + gdy) \cos \lambda x - (gdx - fdy) \sin \lambda x \right] \right\} + \\ &+ \cos \lambda x \cdot \int_\Gamma \left\{ e^{-\int \frac{a^2+b^2}{a} dx} \left[(gdx - fdy) \cos \lambda x + (fdx + gdy) \sin \lambda x \right] \right\} = \\ &= -\alpha(x,y) \sin \lambda x - \beta(x,y) \cos \lambda x\end{aligned} \tag{39}$$

We have:

Theorem 10: If for the coefficient $A(z,\bar{z}) = a + ib$ in the Vekua equation (10) is valid the differential equation (37), where λ is a real constant, then the conditions (13) for u-zeros and (14) for v-zeros, contain harmonic functions $\cos \lambda x$, $\sin \lambda x$ and they are given with (38) and (39). If (f,g) and (α,β) satisfy this conditions, than the conditions (38) and (39) are also equations for determining of the locations of the zeros only of u and only of v, where from we would find the common zeros as well. Here in (38) and (39) we have to switch the solution $y = \Phi(x,C)$ of the equation (37).

Similar attitudes can be formulated for various selections of the six functions $(a,b), (f,g), (\alpha,\beta)$ or for various relations between them.

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