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SOME BOUNDARY VALUE PROBLEMS FOR FINITE INTEGRABLE VEKUA DIFFERENTIAL EQUATIONS

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Abstract. In this paper some boundary value problems for finite integrable Vekua differential equations are considered and solved by reduction to corresponding boundary value problems for analytic functions with known solving procedure.

1. INTRODUCTION

In the well known Vekua monograph [1], the elliptic system of partial differential equations

$$\begin{aligned} u'_x - v'_y &= a(x,y)u + b(x,y)v + f(x,y) \\ u'_y + v'_x &= c(x,y)u + d(x,y)v + g(x,y) \end{aligned} \tag{1.1}$$

is investigated with many details, where $a(x,y), b(x,y), c(x,y), d(x,y), f(x,y)$ and $g(x,y)$ are given continuous real functions of real arguments x and y in a simply connected domain Ω. This system is of great theoretic and practical importance and has many applications in different problems in mechanics. If the second equation is multiplied by i and after addition to the first equation, the following Vekua complex differential equation is obtained

$$U'_{\overline{z}} = MU + N\overline{U} + L \tag{1.2}$$

with the following notations

$$M(z,\bar{z}) = \frac{a+d+ic-ib}{4}, \quad N(z,\bar{z}) = \frac{a-d+ic+ib}{4},$$

$$L(z,\bar{z}) = \frac{f+ig}{2}, \quad U(z,\bar{z}) = u + iv.$$

By the substitution $U = wU_0$, where w is a new unknown function and U_0 is a regular particular solution of the equation $U'_{\overline{z}} = MU$ that can be easily found (see [2]), the equation (1.2) is transformed into the canonical form

$$w'_{\overline{z}} = A\overline{w} + B \tag{1.3}$$

where $A = \frac{N\overline{U}_0}{U_0}$, $B = \frac{L}{U_0}$.
In many cases by using different methods it is possible to find a particular solution

w_0 of (1.3). By using the substitution $w = w_0 + V$, where V is a new unknown function, (1.3) becomes homogeneous equation

$$V'_{\overline{z}} = A\overline{V} \tag{1.4}$$

It was shown by Vekua [1] that the general solution of equation (1.3) is

$$w(z,\overline{z}) = \phi(z) + \iint\limits_T \Gamma_1(z,t)\phi(t)dT + \iint\limits_T \Gamma_2(z,t)\overline{\phi(t)} -$$

$$-\frac{1}{\pi}\iint\limits_T \Omega_1(z,t)B(t)dT - \frac{1}{\pi}\iint\limits_T \Omega_2(z,t)\overline{B(t)}dT \tag{1.5}$$

where

$$\Gamma_1(z,t) = \sum_{j=1}^{\infty} K_{2j}(z,t), \qquad \Gamma_2(z,t) = \sum_{j=1}^{\infty} K_{2j+1}(z,t)$$

$$K_1(z,t) = -\frac{A(t)}{\pi(t-z)}, \qquad K_n(z,t) = \iint\limits_T K_1(z,\sigma)\overline{K_{n-1}(\sigma,t)}dT_\sigma$$

$$\Omega_1(z,t) = \frac{1}{t-z} + \iint\limits_T \frac{\Gamma_1(z,\sigma)}{t-\sigma}dT_\sigma, \quad \Omega_2(z,t) = \iint\limits_T \frac{\Gamma_2(z,\sigma)}{\bar{t}-\bar{\sigma}}dT_\sigma$$

and $\phi(z)$ is an arbitrary analytic function.

In fact, the solution (1.5) practically cannot be used because it contains double singular integrals with Cauchy kernel, which generally cannot be represented in a closed (finite) form. In many physical, mechanical and technical problems, the real and imaginary part of the solution have some special physical meaning. If it is possible to find a general solution of the Vekua complex differential equation in finite, closed and explicit form as

$$w = w(z, \bar{z}, Q(z), \overline{Q(z)}), \tag{1.6}$$

then the real and imaginary part can be separated. But such examples can not be found in literature.

In [3] Čanak has established so called "basic family" for finite integrable Vekua differential equations

$$w'_{\overline{z}} = -\frac{\varphi'(z)}{\varphi + \overline{\varphi}}\overline{w}, \tag{1.7}$$

where $\varphi(z)$ is a given analytic function. He has shown that the general solution of the equation (1.7) is

$$w = Q'(z) - (Q + \overline{Q})\frac{\varphi'(z)}{\varphi + \overline{\varphi}} \tag{1.8}$$

where $Q(z)$ is a class of analytic functions $\mathcal{A}$ that satisfied condition $\overline{Q(z)} = Q(\bar{z})$. Its Taylor series contents only real coefficients that enable conjugations "term by term". So, the relations

$$\overline{Q(z)} = \overline{\sum_{k=0}^{\infty} c_k z^k} = \sum_{k=0}^{\infty} c_k \overline{z^k} = Q(\overline{z})$$

and

$$\overline{Q'(z)} = \overline{\sum_{k=1}^{\infty} kc_k z^{k-1}} = \sum_{k=1}^{\infty} kc_k \overline{z}^{k-1} = Q'(\overline{z}) = \overline{Q}'_{\overline{z}}$$

hold. From the "basic family" (1.7) can be generated new classes of finite integrable Vekua equations.

In the theory of Vekua differential equations the boundary value problems play an important role. The main method of solving these equations is its reduction to corresponding boundary value problems for analytic functions. This reduction usually is very complicated. In this paper through two examples it is shown that procedure for solving boundary value problems for finite integrable Vekua differential equations is significantly easier.

2. A BOUNDARY VALUE PROBLEM FOR FINITE INTEGRABLE VEKUA DIFFERENTIAL EQUATION

J. Čerskii [4] has studied and solved the following

Boundary value problem R: Let the functions $a(x), b(x)$ and $G(x)$ are three continuous real functions in some real domain. The problem is to find an analytic function $Q(z)$ from the upper complex half plane that on the x-axis satisfied the boundary value condition

$$a(x)Q(x) + b(x)Q'(x) = G(x). \tag{2.1}$$

This boundary value problem can be generalized to Vekua complex differential equation.

Boundary value problem V: Let find a solution of the Vekua differential equation (1.7) that on the x-axis satisfied the boundary value condition

$$a(x)w(x) + b(x)w'_{\overline{z}}(x) = G(x). \tag{2.2}$$

Solution. The general solution of the equation (1.7) is (1.8) and the condition

$$w'_{\overline{z}} = -\overline{Q'}_{\overline{z}} \frac{\varphi'(z)}{\varphi + \overline{\varphi}} + (Q + \overline{Q}) \frac{\varphi'(z)\overline{\varphi'(z)}}{(\varphi + \overline{\varphi})^2}$$

is satisfied. On the x-axis the conditions

$$w(x) = Q'(x) - Q(x)\frac{\varphi'(x)}{\varphi(x)}, \quad \left(\varphi(x) = \overline{\varphi(x)};\ Q(x) = \overline{Q(x)};\ \varphi, Q \in \mathcal{A}'\right),$$

and

$$w'_{\overline{z}}(x) = -Q'(x)\frac{\varphi'(x)}{2\varphi(x)} + Q(x)\frac{\varphi'^2(z)}{2\varphi^2(x)} \tag{2.3}$$

are valid also. By substitution of (2.3) in boundary value condition (2.2), the relation

$$a(x)Q'(x) - a(x)Q(x)\frac{\varphi'(x)}{\varphi(x)} - b(x)Q'(x)\frac{\varphi'(x)}{2\varphi(x)} + b(x)Q(x)\frac{\varphi'^2(x)}{2\varphi^2(x)} = G(x)$$

is obtained and after some arranging it becomes

$$\left[\frac{b(x)\varphi'^2(x)}{2\varphi^2(x)} - \frac{a(x)\varphi'(x)}{\varphi(x)}\right] Q(x) + \left[a(x) - \frac{b(x)\varphi'(x)}{2\varphi(x)}\right] Q'(x) = G(x). \tag{2.4}$$

The condition (2.4) is boundary value problem R for determining the unknown analytic function Q(x) that has a known solving procedure. By determining the function $Q(x)$ and its substitution in (1.8), the solution of the boundary value problem V is obtained.

The problem of solving boundary value problem V and number of its linear independent solutions is it's reduction to the correspondent boundary value problem (2.4) for analytic functions.

Example 1. Solve the Vekua differential equation

$$w'_{\overline{z}} = -\frac{1}{z+\overline{z}}\overline{w} \tag{2.5}$$

which solution on the x-axis satisfy the following boundary value condition

$$w(x) + xw'_{\overline{z}}(x) = \frac{x}{2}. \tag{2.6}$$

Solution. According (1.8), the general solution of (2.5) is

$$w(z,\overline{z}) = Q'(z) - \frac{Q+\overline{Q}}{z+\overline{z}}, \quad (Q(z) \in \mathcal{A}') \tag{2.7}$$

and the condition

$$w'_{\overline{z}} = -\frac{\overline{Q}'}{z+\overline{z}} + \frac{Q+\overline{Q}}{(z+\overline{z})^2}$$

is satisfied. The relations $w(x) = Q'(x) - \frac{Q(x)}{x}$ and $w'_{\overline{z}}(x) = -\frac{Q'(x)}{2x} + \frac{Q(x)}{2x^2}$ are valid on the x-axis also, and by these substituting in (2.6) the following

$$Q'(x) - \frac{Q(x)}{x} = x \tag{2.8}$$

is obtained. The condition (2.6) is the boundary value problem R for determining the analytic function $Q(z)$. The solution of this problem is $Q(z) = z^2 + cz$, where c is an arbitrary constant. By it's substitution in (2.7) the solution of boundary value problem (2.5)-(2.6) is obtained and it is given by the relation

$$w(z,\overline{z}) = 2z + c - \frac{z^2 + cz + \overline{z}^2 + c\overline{z}}{z+\overline{z}} \tag{2.9}$$

3. ON A QUASI-VEKUA DIFFERENTIAL EQUATION

Let observe generalized Vekua differential equation

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = Bw + C\bar{w} + D, \tag{3.1}$$

where the coefficients $A = A(z,\bar{z}),\ B = B(z,\bar{z}),\ C = C(z,\bar{z}),\ D = D(z,\bar{z})$ are given continuous functions in some domain Ω. This equation is solved by iterations [5]. In the case when the coefficient $A(z,\bar{z}) = 0$, then (3.1) represents common Vekua differential equation (1.2).

According to the basic principle that the general solution of a complex differential equation of the first order contents one arbitrary analytic function, the following definition is given

Definition 3.1. *The function $w = w(z, \bar{z}, Q(z))$ is F-general solution of the generalized Vekua differential equation (3.1) in some domain Ω if it is continuous and differentiable on $\bar{z}$ in the same domain and identically satisfied the equation.*

The function w contains an arbitrary analytic function $Q(z)$, but it can contain the functions $\overline{Q(z)}$ and $Q'(z)$ also.

Definition 3.2. *The generalized Vekua differential equation (3.1) is F-finite integrable function if there exists F-general solution in a finite and closed form $w = w(z, \bar{z}, Q(z))$.*

Let in the differential equation (3.1) introduce the substitution

$$w + A\bar{w} = f, \tag{3.2}$$

where $f = f(z, \bar{z})$ is a new unknown function. By conjugation, (3.2) becomes

$$\bar{w} + \bar{A}w = \overline{f}. \tag{3.3}$$

The solution of the system (3.2)-(3.3) is

$$w = \frac{A\overline{f} - f}{A\overline{A} - 1}. \tag{3.4}$$

Substituting (3.4) in (3.1) and after some arranging the equation (3.1) becomes common Vekua equation

$$f'_{\bar{z}} = \frac{(C + A'_{\bar{z}})\,\overline{A} - B}{A\overline{A} - 1} f + \frac{BA - (C + A'_{\bar{z}})}{A\overline{A} - 1} \overline{f} + D. \tag{3.5}$$

In [2] Fempl showed that the general solution of the degenerate Vekua differential equation

$$U'_{\bar{z}} = MU + L \tag{3.6}$$

is

$$U(z, \bar{z}) = e^{I(M)} \left[Q(z) + ILe^{-J(M)} \right] \tag{3.7}$$

where I is an integral operator inverse to the differential operator $\partial/\partial\bar{z}$. In (3.7) the term $\bar{U}$ that is characteristic for common Vekua equation is missing. So the equation (3.5) is finite-integrable iff the coefficient in front of $\overline{f}$ is annulated and the following theorem can be formulated

Theorem 3.1. *The generalized Vekua differential equation (3.1) is a finite integrable iff*

$$BA - C - A'_{\bar{z}} = 0. \tag{3.8}$$

Remark 3.1: If the condition (3.8) is satisfied, then the equation (3.5) is reduced to

$$f'_{\bar{z}} = Bf + D, \tag{3.9}$$

and the equation (3.1) is reduced to

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = Bw + (BA - A'_{\bar{z}})\,\bar{w} + D. \tag{3.10}$$

So, the general solution of (3.9) can be obtained on the base of the formula (3.7) and it's substitution in (3.4) leads to F-general solution of the equation (3.10).

Remark 3.2: The complex differential equation (3.10) is not a generalization of the common Vekua equation (1.2) because for $A = 0$ the coefficient in front of $\bar{w}$ is annulated. That is the reason why the equation (3.10) is named quasi Vekua differential equation and it is an important F-finite integrable case of generalized Vekua equation (3.1).

4. HILBERT BOUNDARY VALUE PROBLEM FOR QUASI VEKUA DIFFERENTIAL EQUATION

The determination of a general solution of the common or generalized Vekua differential equation in a finite explicit form makes the solving process of many boundary value problems easier. It is especially important in the case when the real and imaginary part of solution can be separated. To illustrate this fact, Hilbert boundary value problem for quasi Vekua differential equation will be solved. This solution has an application and physical interpretation in the theory of elastic shells [1].

Gahov in his monograph [6] formulated the following Hilbert boundary value problem for analytic functions

Boundary value problem H_A**:** Let L be a simply, smooth and closed contour that restricts the domain Ω. Let $a(t), b(t)$ and $c(t), (t \in L)$ be given functions that on the contour L satisfy Hölder condition of continuity. The problem is to find a function of the type $f(z) = u(x, y) + iv(x, y)$ which is an analytic in Ω and continuous on L and satisfis Hölder boundary value problem

$$a(t)u(t) + b(t)v(t) = c(t). \tag{4.1}$$

Gahov showed that the solution of the boundary value problem (4.1) is

$$f(z) = \begin{cases} e^{i\gamma(z)} \left[S|t|^{-\chi} e^{\omega_1(t)} c(t) + i\beta_0 \right], & \chi = 0 \\ z^{\chi} e^{i\gamma(z)} \left[S|t|^{-\chi} e^{\omega_1(t)} c(t) + Q(z) \right], & \chi > 0 \\ z^{\chi} e^{i\gamma(z)} \left[S|t|^{-\chi} e^{\omega_1(t)} c(t) + iC \right], & \chi < 0. \end{cases} \tag{4.2}$$

In (4.2) the notations have the following meaning:
χ-index of Hilbert boundary value problem;
S-Schwarz operator;
$\gamma(z) = \omega(x, y) + i\omega_1(x, y)$;
$Q(z) = i\beta_0 + \sum\limits_{k=1}^{\chi} \left(c_k z^k - \overline{c_k} z^{-k} \right)$.

On the base of the known facts, it is possible to solve also Hilbert boundary value problem for quasi Vekua differential equation.

Boundary value problem H_{QV}**:** Let L be a simply, smooth and closed contour that restricts the domain Ω. Let $\mathcal{A}(t), \mathcal{B}(t)$ and $\mathcal{C}(t), (t \in L)$ be given functions that on the contour L satisfy Hölder condition of continuity. The problem is to find a regular solution of quasi Vekua equation (3.10) that on contour L satisfy boundary value problem

$$\mathcal{A}(t)u(t)+\mathcal{B}(t)v(t)=\mathcal{C}(t),\ (t\in L). \tag{4.3}$$

Solution. By using the new notations

$$M=e^{I(B)},\quad N=e^{I(B)}IDe^{-I(B)}$$

the solution (3.7) becomes

$$\begin{aligned} w(z,\bar z)&=\frac{A\overline{f}-f}{A\overline{A}-1}=\frac{A\left(\overline{MQ}+\overline{N}\right)-MQ-N}{A\overline{A}-1}=\\ &=-\frac{M}{A\overline{A}-1}Q+\frac{A\overline{M}}{A\overline{A}-1}\overline{Q}+\frac{A\overline{N}-N}{A\overline{A}-1}=\mathcal{M}Q+\mathcal{N}\overline{Q}+\mathcal{P}, \\ \mathcal{M}&=-\frac{M}{A\overline{A}-1},\quad \mathcal{N}=\frac{A\overline{M}}{A\overline{A}-1},\quad \mathcal{P}=\frac{A\overline{N}-N}{A\overline{A}-1}. \end{aligned} \tag{4.4}$$

Separating the real and imaginary part of the function $w=u+iv$, the following is obtained

$$\begin{aligned} w&=u+iv=(m_1+im_2)(q_1+iq_2)+(n_1+in_2)(q_1-iq_2)+p_1+ip_2=\\ &=(m_1q_1-m_2q_2+n_1q_1+n_2q_2+p_1)+i(m_1q_2+m_2q_1-n_1q_2+n_2q_1+p_2) \end{aligned}$$

or

$$\begin{aligned} u&=m_1q_1-m_2q_2+n_1q_1+n_2q_2+p_1\\ v&=m_1q_2+m_2q_1-n_1q_2+n_2q_1+p_2. \end{aligned} \tag{4.5}$$

Substituting the values (4.5) in boundary value problem (4.3) the relation

$$\mathcal{A}(t)(m_1q_1-m_2q_2+n_1q_1+n_2q_2+p_1)+\mathcal{B}(t)(m_1q_2+m_2q_1-n_1q_2+n_2q_1+p_2)=\\ =\mathcal{C}(t)$$

or

$$q_1[\mathcal{A}(m_1+n_1)+\mathcal{B}(m_2+n_2)]+q_2[\mathcal{A}(-m_2+n_2)+\mathcal{B}(m_1-n_1)]=\mathcal{C}-\mathcal{A}p_1-\mathcal{B}p_2 \tag{4.6}$$

is obtained. The relation (4.6) is Hilbert boundary value problem for determination of the analytic function $Q=q_1+iq_2$. If this function is determinate by the formula (4.2) and then by its substitution in (4.4), the solution of the Hilbert boundary value problem for quasi Vekua differential equation (3.10) will be obtained.

The question of solvable and the number of linearly independent solutions of the boundary value problem (3.10)-(4.3) is reduced to the corresponding boundary value problem for analytic functions and depends of the index χ of the boundary value problem that has the following value

$$\chi=Ind\left[\mathcal{A}(t)+i\mathcal{B}(t)\right]=\frac{1}{2\pi}\left[\arg\left(\mathcal{A}(t)+i\mathcal{B}(t)\right)\right]_L \tag{4.7}$$

Example. Find a solution of the complex differential equation

$$w'_{\bar z}+2\bar w'_{\bar z}=\frac{1}{\bar z}w+\frac{2}{\bar z}\bar w+\frac{1}{\bar z},\quad (w=u+iv), \tag{4.8}$$

that on the unit circle satisfies Hilbert boundary value problem

$$\left(3\cos 2t - \frac{3}{2}\sin t\right)u(t) + \left(\sin 2t + \frac{1}{2}\cos t\right)v(t) = \frac{1}{4} + \frac{1}{2}\sin t. \tag{4.9}$$

Solution. By the substitution $w + 2\bar{w} = f$, or by

$$w = \frac{2\overline{f} - f}{3}, \tag{4.10}$$

the equation (4.8) is transformed into the complex differential equation

$$f'_{\bar{z}} = \frac{f}{\bar{z}} + \frac{1}{\bar{z}}. \tag{4.11}$$

According (3.7), it's solution is

$$f = \bar{z}Q(z) - 1, \tag{4.12}$$

where $Q(z)$ is an arbitrary analytic function. Substituting (4.12) in (4.10), the general solution of quasi Vekua differential equation (4.8) is obtained and it is

$$w(z,\bar{z}) = \frac{2z\overline{Q(z)} - \bar{z}Q(z) - 1}{3}. \tag{4.13}$$

Separating the real and imaginary part of the solution (4.13)

$$w = \frac{xq_1 + yq_2 - 1}{3} + i\left(-xq_2 + yq_1\right)$$

is obtained that on the unit circle is

$$\begin{aligned} u\,|_L &= \left(q_1\cos t + q_2\sin t - 1\right)/3 \\ v\,|_L &= q_1\sin t - q_2\cos t. \end{aligned} \tag{4.14}$$

By a substitution of (4.14) in boundary value condition (4.9), Hilbert boundary value problem

$$\cos t \cdot q_1(t) - \left(\sin t + \frac{1}{2}\right)q_2(t) = \cos 2t + \frac{1}{4} \tag{4.15}$$

is obtained and it is used for determination of the analytic function $Q(z) = q_1(x,y) + iq_2(x,y)$. After some calculations, the unique solution of the boundary value problem (4.15) is found in the form of

$$Q(z) = z - \frac{1}{2}i.$$

By it's substitution in (4.13), the unique solution

$$w(z,\bar{z}) = \frac{z\bar{z} + iz + (i/2)\bar{z} - 1}{3}$$

of the boundary value problem (4.8)-(4.9) is obtained.

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