Математички Билтен
17 (XLIII)
1993 (53-62)
Скопје, Македонија

ISSN 0351-336X

ON VECTOR SUBBUNDLES OF THE VECTOR BUNDLES

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Abstract. In this paper are proved some theorems which give necessary and sufficient conditions for a real vector bundle of rank k to admit vector subbundle of rank m ($m\le k$), or to admit m linearly independent vector fields. In the last section are considered complex vector bundles. Some ideas from the theory of complex commutative vector valued groups [2] are introduced in the complex vector bundles and two theorems are proved.

1. Introduction. Let $\xi=(E,\pi,B_n)$ be a real vector bundle of rank k. It can be considered as an associated bundle of the locally trivial principal $GL(k;\mathbb{R})$-bundle with a bundle $\mathbb{R}^k$. Let $\{U_\alpha\}$ be a locally trivial cover of B_n and let $\phi_\alpha\colon U_\alpha\times\mathbb{R}^k\to E_{U_\alpha}=\pi^{-1}(U_\alpha)$ be the corresponding homeomorphisms. The corresponding transition functions are

$$\phi_{\beta\alpha}\colon U_\alpha\cap U_\beta\to GL(k;\mathbb{R})$$
$$\phi_{\beta\alpha}(b)=\phi_{\beta,b}^{-1}\cdot\phi_{\alpha,b}\colon \mathbb{R}^k\to\mathbb{R}^k.$$

These functions are continuous and satisfy

$$\phi_{\beta\alpha}^{-1}=\phi_{\alpha\beta}\ \text{on}\ U_\alpha\cap U_\beta$$

and

$$\phi_{\gamma\beta}\cdot\phi_{\beta\alpha}=\phi_{\gamma\alpha}\ \text{on}\ U_\alpha\cap U_\beta\cap U_\gamma$$

for each indices α,β,γ for which $U_\alpha\cap U_\beta\neq\emptyset$ and $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$ respectively.

We say that the vector bundle ξ is reducibled to the group $G\subseteq GL(k;\mathbb{R})$ if there exists a family of trivializing charts $\{(U_\alpha,\phi_\alpha)\}$ such that $\phi_{\beta\alpha}(b)\in G$, for each $b\in U_\alpha\cap U_\beta$ and arbitrary indices α and β. For example, if ξ can be reduced to the group $GL^+(k;\mathbb{R})$ then ξ is called orientabled vector bundle, and if ξ can be reduced to the group $O(n)$, then ξ is called metrizabled vector bundle. Although each vector bundle ξ is always metrizabled on a paracompact manifold B_n, in general case ξ may not be orientabled vector bundle.

In section 2. we will give some necessary and sufficient conditions for the vector bundle ξ to admit m linearly independent vector fields. First we give some known results.

Let $k=k_1+k_2$, and let us denote by $GL(k_1,k_2;\mathbb{R})$ the following subgroup

$$GL(k_1,k_2;\mathbb{R}) = \{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid A\in GL(k_1;\mathbb{R}),\ B\in GL(k_2;\mathbb{R})\} \quad (1)$$

of $GL(k;\mathbb{R})$. The following theorem can be found in [1] page 122.

Theorem 1. The vector bundle ξ of rank $k=k_1+k_2$ is reducibled to the group $GL(k_1,k_2;\mathbb{R})$ iff ξ is a Whitney's sum of vector subbundles ξ_1 and ξ_2 of ranks k_1 and k_2 respectively.

Obviously, the vector bundle ξ admits a continuous vector field iff ξ can be written as a Whitney's sum of vector bundles ξ_1 and ξ_2 of ranks 1 and k-1 respectively, and ξ_1 is an orientable subbundle. According to the theorem 1 it is possible iff ξ can be reduced to the following group

$$G = \{\left[\begin{array}{c|c} a & 0 \\ \hline 0 & A \end{array}\right] \mid a\in\mathbb{R}^+ \text{ and } A\in GL(k-1;\mathbb{R})\}. \quad (2)$$

By repeating this procedure m times we get the following

Theorem 2. The vector bundle ξ of rank k admits m linearly independent vector fields iff ξ can be reduced to the following subgroup

$$G = \{\left[\begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \\ \hline & 0 & & A \end{array}\right] \mid a_1,\dots,a_m > 0,\ A\in GL(k-m;\mathbb{R})\} \quad (3)$$

of $GL(k;\mathbb{R})$.

Further in section 3. we will consider a special structures on complex vector bundles which are induced by the com(m+k,m)-groups on $\mathbb{C}$ [2].

2. Some results about subbundles of vector bundles.

First we will generalize the theorem 2 by the following

Theorem 3. Let B_n be a paracompact manifold. The vector bundle $\xi=(E,\pi,B_n)$ of rank k admits m linearly independent vector fields iff ξ can be reduced to the following subgroup

$$G = \left\{ \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & \\ 0 & a_{22} & \dots & a_{2m} & B \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & a_{mm} & \\ \hline & & 0 & & C \end{array}\right] \middle| \; a_{ii} > 0 \text{ for } 1 \le i \le m, \right.$$

$$\left. a_{ij} \in \mathbb{R} \text{ for } 1 \le i < j \le m,\ C \in GL(k-m;\mathbb{R}),\ B \in M(m,k-m;\mathbb{R}) \right\} \tag{4}$$

of $GL(k;\mathbb{R})$.

Proof. Let us suppose that the vector bundle ξ reduces to the group G. Since B_n is a paracompact manifold, there exists locally finite open cover with trivializing coordinate neighborhoods $\{U_\alpha\}$, and let $\{f_\alpha\}$ be the decomposition of the unit with respect to the cover $\{U_\alpha\}$. Using the local coordinates, in each neighborhood U_α we define m vector fields as follows

$$\begin{aligned} X^\alpha_{(1)} &= (1,0,\dots,0) \in \mathbb{R}^k \\ X^\alpha_{(2)} &= (0,1,0,\dots,0) \in \mathbb{R}^k \\ &\dots\dots\dots\dots\dots\dots \\ X^\alpha_{(m)} &= (\underbrace{0,\dots,0}_{m-1},1,0,\dots,0) \in \mathbb{R}^k. \end{aligned} \tag{5}$$

Further we define m global vector fields on ξ as follows

$$X_{(r)} = \sum_\alpha X^\alpha_{(r)} f_\alpha, \quad 1 \le r \le m. \tag{6}$$

In order to prove that these vectors are linearly independent, it is sufficient to verify it in an arbitrary coordinate neighborhood U_α. Using (4) and (5) we obtain

$$X^\beta_{(r)} = (b^\beta_{1r},\dots,b^\beta_{rr},0,\dots,0), \quad 1 \le r \le m$$

with respect to the coordinate neighborhood U_β, where $b^\beta_{rr} > 0$. Using that $f_\beta \ge 0$ and $\sum_\beta f_\beta = 1$, it follows that the vector fields $X_{(1)},\dots,X_{(m)}$ are linearly independent in U_β and hence in B_n.

Conversely, let the vector bundle ξ admits m linearly independent vector fields. Then ξ can be reduced to the group G given by (3), and hence can be reduced to the group given by (4). □

Theorem 4. Let B_n be a paracompact differentiable manifold.

a) The vector bundle $\xi=(E,\pi,B_n)$ admits a vector subbundle of rank m iff ξ can be reduced to the following subgroup

$$G = \{\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mid A \in GL(m;\mathbb{R}),\ C \in GL(k-m;\mathbb{R}),\ B \in M(m,k-m;\mathbb{R})\} \qquad (7)$$

of $GL(k;\mathbb{R})$.

b) The vector bundle $\xi=(E,\pi,B_n)$ admits an orientabled vector subbundle with rank m iff ξ can be reduced to the following subgroup

$$G = \{\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mid A \in GL^+(m;\mathbb{R}),\ C \in GL(k-m;\mathbb{R}),\ B \in M(m,k-m;\mathbb{R})\} \qquad (8)$$

of $GL(k;\mathbb{R})$.

Proof. a) Let us suppose that ξ can be reduced to the group G given by (7). Then we define vector subbundle on $\xi|_{U_\alpha}$ which is generated by

$$X_{(1)} = (1,0,\dots,0) \in \mathbb{R}^k$$
$$X_{(2)} = (0,1,0,\dots,0) \in \mathbb{R}^k$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$X_{(m)} = (\underbrace{0,\dots,0}_{m-1},1,0,\dots,0) \in \mathbb{R}^k$$

in each trivializing coordinate neighborhood U_α. It is easy to verify that this subbundle does not depend on the neighborhood U_α, because the subspace of $\mathbb{R}^k$ which is generated by $X_{(1)},\dots,X_{(m)}$ is invariant under the matrices of G which are defined by (7). Thus, we have defined a global vector subbundle of ξ with rank m.

The converse is a consequence from the Theorem 1.

b) This is a consequence of a). □

Further we will prove two theorems where the transition functions $(\phi_{\alpha\beta})$ belong only to a set of matrices G which is not a group. For example G can be the set of positive definite kxk matrices.

Theorem 5. Let B_n be a paracompact manifold. The vector bundle $\xi=(E,\pi,B_n)$ of rank k admits m linearly independent vector fields iff there exists a system of trivializing charts $\{(U_\alpha,\phi_\alpha)\}$ such that the matrices $\phi_{\alpha\beta}(b)$ belong to the set

$$G = \{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq 0, \text{ A is positive definite } m\times m \text{ matrix, } B\in M(m,k-m;\mathbb{R}),\ C\in M(k-m,m;\mathbb{R}),\ D\in M(k-m,k-m;\mathbb{R})\} \quad (9)$$

for each indices α and β such that $U_\alpha \cap U_\beta \neq \emptyset$, and each $b\in U_\alpha \cap U_\beta$.

Proof. Let us suppose that there exists a system of trivializing charts $\{(U_\alpha,\phi_\alpha)\}$ such that $\phi_{\alpha\beta}(b)\in G$. Let $\{f_\alpha\}$ be a decomposition of the unit, which corresponds to a locally finite subcover $\{U'_\alpha\}$. In each trivializing coordinate neighborhood U'_α we define m vector fields by

$$\begin{aligned} X^\alpha_{(1)} &= (1,0,\dots,0)\in\mathbb{R}^k \\ X^\alpha_{(2)} &= (0,1,0,\dots,0)\in\mathbb{R}^k \\ &\dots\dots\dots\dots \\ X^\alpha_{(m)} &= (\underbrace{0,\dots,0}_{m-1},1,0,\dots,0)\in\mathbb{R}^k. \end{aligned}$$

Now we define m global vector fields

$$X_{(r)} = \sum_\alpha f_\alpha X^\alpha_{(r)}, \quad 1 \le r \le m.$$

We choose an arbitrary chart (U'_β,ϕ'_β), and we will prove that $X_{(1)},\dots,X_{(m)}$ are linearly independent vector fields with respect to that coordinate system. Indeed, in that coordinate system it is

$$\begin{aligned} X^\beta_{(1)} &= (A^\beta_{11},A^\beta_{21},\dots,A^\beta_{k1}) \\ &\dots\dots\dots\dots \\ X^\beta_{(m)} &= (A^\beta_{1m},A^\beta_{2m},\dots,A^\beta_{km}) \end{aligned}$$

where

$$A^\beta = \begin{bmatrix} A^\beta_{11} & \dots & A^\beta_{1m} \\ \dots & \dots & \dots \\ A^\beta_{m1} & \dots & A^\beta_{mm} \end{bmatrix}$$

is a positive definite $m\times m$ matrix. Using that $f_\beta \ge 0$ and $\sum_\beta f_\beta=1$, we obtain

$$\begin{aligned} X_{(1)} &= (A_{11},A_{21},\dots,A_{k1}) \\ &\dots\dots\dots\dots \\ X_{(m)} &= (A_{1m},A_{2m},\dots,A_{km}) \end{aligned}$$

according to the chosen coordinate system, where

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ \dots & \dots & \dots \\ A_{m1} & \dots & A_{mm} \end{bmatrix}$$

is a positive definite matrix. Hence $\det A \neq 0$, and the vectors $X_{(1)}, \dots, X_{(m)}$ are linearly independent.

Conversely, let us suppose that the vector bundle ξ admits m linearly independent vector fields. Theorem 3 implies that ξ can be reduced to the group G which is given by (4). But that group is a subset of the set G which is given by (9). □

We notice that in the Theorem 5, C way be a zero-matrix, and that special case can be compared with the Theorem 4.

Theorem 6. Let B_n be a paracompact manifold. The vector bundle $\xi=(E,\pi,B_n)$ of rank k admits a non-zero vector field iff there exists a system of trivializing charts $\{(U_\alpha,\phi_\alpha)\}$ such that the matrices $\phi_{\alpha\beta}(b)$ belong to the set

$$G = \{[A]_{k\times k} \mid \det A \neq 0, \sum_{i,j=1}^{k} A_{ij} > 0\} \tag{10}$$

for each indices α and β such that $U_\alpha \cap U_\beta \neq \emptyset$ and each $b \in U_\alpha \cap U_\beta$.

Proof. Let ξ admits a vector field X which is non-zero at each point. In a neighborhood of each point, the coordinate system can be chosen such that X has coordinates $(1,1,\dots,1) \in R^k$. Now we will prove that the transition matrices are such that $\sum_{1\le i<j\le k} A_{ij} > 0$. Indeed, these matrices left the vector $(1,1,\dots,1) \in R^k$ invariant, i.e. $\sum_{j=1}^{k} A_{ij} = 1$, and hence $\sum_{i,j=1}^{k} A_{ij} = k > 0$.

Conversely, let us suppose that there exist trivializing charts $\{(U_\alpha,\phi_\alpha)\}$ such that $\phi_{\alpha\beta}(b)$ belongs to the set G given by (10), for each indices α and β such that $U_\alpha \cap U_\beta \neq \emptyset$ and each $b \in U_\alpha \cap U_\beta$. Since B_n is a paracompact manifold, we can suppose that $\{U_\alpha\}$ is a locally finite subcover, and let $\{f_\alpha\}$ be the corresponding decomposition of the unit. In each chart $\{(U_\alpha,\phi_\alpha)\}$ we define a vector field X^α with local coordinates

$$X^\alpha = (1,1,\dots,1) \in R^k.$$

Now we define a global vector field by

$$X = \sum_\alpha f_\alpha X^\alpha$$

We only have to prove that $X \neq 0$ at each point. It is sufficient to choose an arbitrary chart (U_β, ϕ_β) and to prove that $X \neq 0$ according to it. Indeed, if $U_\alpha \cap U_\beta \neq \emptyset$, then X^α has local coordinates

$$X^\alpha = (p_1^\alpha, \dots, p_k^\alpha)$$

such that $\sum_{i=1}^{k} p_i^\alpha > 0$. Hence it follows that according to that coordinate system, X has coordinates

$$X = \sum_\alpha f_\alpha X^\alpha = (p_1, \dots, p_k)$$

where $\sum_{i=1}^{k} p_i > 0$, and X is a non-zero vector. □

3. One structure on complex vector bundle. The idea of this section is inspired from the theory of commutative vector valued groups [2]. First we give some facts from that theory, which will need us further.

Let $Q^{(m)}$ be the m-th permutation product of Q, i.e. $Q^{(m)} = Q^m/\approx$ where $\approx$ is the equivalence on Q^m which is defined by

$$x_1^m \approx y_1^m \Leftrightarrow x_1, x_2, \dots, x_m \text{ is a permutation of } y_1, \dots, y_m,$$

where we use the notation $t = z_1^m$ instead of $(z_1, \dots, z_m)$.

Definition 1. A map $f\colon Q^{(n)} \to Q^{(m)}$, $(n-m=k\geq 1)$ is called a com(n,m)-group if the following two axioms are satisfied

(i) (associativity)

For each $1 \leq i \leq k$ and each $x_1^{n+k} \in Q^{(n+k)}$

$$f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}), \tag{11}$$

(ii) (solvability)

For each $a \in Q^{(k)}$ and $b \in Q^{(m)}$ the equation

$$f(ax) = b \tag{12}$$

has solution $x \in Q^{(k)}$.

One of the most researched class of such structures is the class of affine com(m+k,m)-groups, and moreover each locally euclidean and connected topological com(m+k,m)-group which is known until now is isomorphic to an affine com(m+k,m)-group. They are defined on $Q=\mathbb{C}\setminus\{a_1,\dots,a_t\}$ $(0\le t\le m)$, where $a_1,\dots,a_t$ are distinct complex numbers. In that case $Q^{(m)}$ is homeomorphic to

$$(\mathbb{C}\setminus\{0\})^t\times\mathbb{C}^{m-t}. \tag{13}$$

It is interesting to mention here that if (M,f) is a locally euclidean topological com(m+k,m)-group, then dimM=2. We also mention that each com(m+k,m)-group induces a usual commutative group $(M^{(m)},*)$. For a given commutative group $(M^{(m)},*)$ we want to obtain a com(m+k,m)-group. It can be done as follows.

We say that the commutative group $(M^{(m)},*)$ has the property P if there exists a subset $S\subseteq M^{(m)}$ such that for each $y\in M^{(m)}$, there exists unique $x_1^m\in S^{(m)}$ such that

$$y = x_1*\dots*x_m.$$

Now it is easy to verify that if $(M^{(m)},*)$ has the property P, then it defines a com(m+k,m)-group (S,f), where

$$f(x_1^{m+k}) = y_1^m \text{ iff } x_1*\dots*x_{m+k} = y_1*\dots*y_m. \tag{14}$$

Specially, if $M=\mathbb{C}$, then $M^{(m)}=\mathbb{C}^m$ and $*$ is the addition. The affine com(m+k,m)-groups are determined by the following smooth surfaces S:

$$S = \tau(\lambda+S_0) \tag{15}$$

where $S_0=\{(z,z^2,\dots,z^m)\mid z\in\mathbb{C}\}$, $\lambda\in\mathbb{C}^m$ and τ is an automorphism of $(\mathbb{C}^m,+)$. We mention that until now we do not know any other smooth surface $S\subseteq\mathbb{C}^{(m)}=\mathbb{C}^m$, except those described by (15).

Now we return to the vector bundles. Having all that in mind, we give the following two definitions.

<u>Definition 2</u>. Let $\xi=(E,\pi,B_n)$ be a complex vector bundle with complex dimension m. We say that the vector bundle has the property P, if for each point $p\in B_n$, $\pi^{-1}(p)$ as a complex vector bundle admits a smooth subset $S(p)\subseteq\pi^{-1}(p)$ such that

i) For each $X\in\pi^{-1}(p)$ there exists unique $X_1^m\in S(p)^{(m)}$ such that

$$X = X_1 +\dots+ X_m,$$

ii) The distribution $S(p)$ depends continuously on p.

Definition 3. Let $\xi=(E,\pi,B_n)$ be a complex vector bundle with complex dimension m, and let the vector bundle has the property P. If the corresponding subsets $\{S(p)\}$ have the following form

$$S(p) = \tau_p(\lambda_p+S_o) \tag{16}$$

with respect to a chosen coordinate system, where $S_o=\{(z,z^2,\dots,z^m) \mid z\in\mathbb{C}\}$, $\lambda_p\in\mathbb{C}^m$ and τ_p is an automorphism on $\mathbb{C}^m$, then we say that the vector bundle has the affine property AP.

If the vector bundle ξ has the property P, then it admits a continuous family of com(m+k,m)-groups $(S(p),f_p)$ for $p\in B_n$, and if the vector bundle ξ has the affine property AP, then it admits a continuous family of affine com(m+k,m)-groups $(S(p),f_p)$ for $p\in B_n$.

We mention that the group of automorphisms of $\mathbb{C}^m$ is isomorphic to $GL(2m;\mathbb{R})$ but not to $GL(m;\mathbb{C})$. Further we will prove two theorems about the vector bundles with the affine property AP.

Theorem 7. If the complex vector bundle ξ with a complex dimension m can be written as a Whitney's sum of m complex 1-dimensional vector bundles $\xi_1,\dots,\xi_m$, i.e.

$$\xi = \xi_1\oplus\xi_2\oplus\dots\oplus\xi_m, \tag{17}$$

then ξ has the affine property AP.

Proof. Let us suppose that (17) holds. We choose the coordinate system in $\pi^{-1}(p)$ such that

$$\xi_1 = \{(z,0,\dots,0) \mid z\in\mathbb{C}\}$$
$$\xi_2 = \{(0,z,0,\dots,0) \mid z\in\mathbb{C}\}$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\xi_m = \{(0,\dots,0,z) \mid z\in\mathbb{C}\}.$$

Then we define

$$S(p) = \{(z,z^2,\dots,z^m) \mid z\in\mathbb{C}\}$$

with respect to that coordinate system. Thus ξ has the affine property AP. □

Theorem 8. If ξ is a vector bundle with the affine property AP, then there exist m real 2-dimensional vector bundles $\xi_i^{\mathbb{R}}$ $(1 \le i \le m)$, such that

$$\xi^{\mathbb{R}} = \xi_1^{\mathbb{R}} \oplus \ldots \oplus \xi_m^{\mathbb{R}} \tag{18}$$

where $\xi^{\mathbb{R}}$ is considered as a real 2m-dimensional vector bundle.

<u>Proof</u>. Let $\{S(p) \subseteq \pi^{-1}(p)\}$ be the family of subsets of $\pi^{-1}(p)$. For each p, there exist λ_p and τ_p such that $S(p)=\tau_p(\lambda_p+S_0)$ where $S_0=\{(z,z^2,\ldots,z_m) \mid z\in\mathbb{C}\}$. Since $\tau_p\in GL(2m;\mathbb{R})$, we define

$$\xi_1^{\mathbb{R}} = \tau_p(\{(z,0,\ldots,0) \mid z\in\mathbb{C}\})$$
$$\xi_2^{\mathbb{R}} = \tau_p(\{(0,z,0,\ldots,0) \mid z\in\mathbb{C}\})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\xi_m^{\mathbb{R}} = \tau_p(\{(0,\ldots,0,z) \mid z\in\mathbb{C}\}).$$

Then $\xi_1^{\mathbb{R}},\ldots,\xi_m^{\mathbb{R}}$ do not depend on the coordinate system, and

$$\xi^{\mathbb{R}} = \xi_1^{\mathbb{R}} \oplus \ldots \oplus \xi_m^{\mathbb{R}}. \ \square$$

R E F E R E N C E S

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ЗА ВЕКТОРСКИ ПОДРАСЛОЈУВАЊА ОД ВЕКТОРСКИ РАСЛОЈУВАЊА

К. Тренчевски

Р е з и м е

Во овој труд се докажани неколку теореми кои даваат потребен и доволен услов за да едно реално векторско раслојување од ранг k допушта векторско подраслојување со ранг m ($m \le k$), односно да допушта m линеарно независни векторски полиња. Во последниот параграф се разгледуваат комплексни векторски раслојувања. Се дефинира кога едно комплексно векторско раслојување има особина P, односно афина особина AP, а потоа се докажани две теореми во врска со раслојувањата со афина особина AP. Резултатите од овој параграф се инспирирани од теоријата на комутативните векторско вредносни групи [2].

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