Математички Билтен ISSN 0351–336X
24 (L)
2000 (57–60)
Скопје, Македонија

SOME INEQUALITIES WITH FACTORIALS REVISITED

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Abstract

In this note some inequalities for finite sums and products with factorials are (im-) proved.

In [1] Z. F. Starc proved several inequalities for sums and products with factorials.

His method mainly relies on the following idea:

If $a_1, a_2, \ldots, a_n$ are positive numbers satisfying $a_1 + a_2 + \cdots + a_n = b$ then there holds

$$b \geq n + \ln(a_1 \cdot a_2 \cdot \ldots \cdot a_n), \quad \text{i.e.} \quad a_1 \cdot a_2 \cdot \ldots \cdot a_n \leq e^{b-n}.$$

We now show that these inequalities are weaker than the classical arithmetic – geometric means – inequality, saying

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n \leq \left(\frac{b}{n}\right)^n, \tag{1}$$

with equality iff all the $a_i's$ are equal.

In order to verify this claim we have to prove the inequality

$$\left(\frac{b}{n}\right)^n \leq e^{b-n}. \tag{2}$$

Let us put $w := \dfrac{b}{n}$.

Then (2) reads $w \leq e^{w-1}$ where $w > 0$.

Upon taking logarithm we note that this inequality is equivalent to the known one $1 + \ln w \leq w$ where $w > 0$.

As a first remark we note that the following result of J. E. Pečarić [3] and used for the proof of [1, Theorem 1] is *incorect.* It says

$$\Gamma(x_1)\cdot\ldots\cdot\Gamma(x_n)=\sqrt{\left(\Gamma(x_1)\cdot\Gamma(x_n)\right)^n} \tag{3}$$

whenever $x_1, x_2, \ldots, x_n$ is a positive increasing arithmetic sequence. For let as an example $n = 3$ and put $x_k = k$, $k = 1,2,3$. Then LHS(3) $= 0!\cdot 1!\cdot 2! = 2$, whereas RHS(3) $= \sqrt{(0!\cdot 2!))^3} = 2\sqrt{2}$. Hence Theorem 1 and its Corollary have to be revised. We now state

Theorem 1′. Let m, n and p be entire numbers such that $m \geq 1$, $n \geq 2$ and $p \geq -m$.

Then for all increasing convex functions $f\colon[1,\infty)\to R$ it holds

$$f\big((m+p)!\big)+f\big((2m+p)!\big)+\cdots+f\big((mn+p)!\big) > nf\left(n\Big(\frac{n+1}{2}\,m+p\Big)!\right). \tag{4}$$

Proof. Using the log-convexity of $\Gamma(z)$ (see e.g. [2, item 3.6.48]) we infer via the geometric – arithmetic means – inequality

$$(m+p)!+(2m+p)!+\cdots+(nm+p)! >$$
$$> n[(m+p)!\cdot(2m+p)!\cdot\ldots\cdot(nm+p)!]^{1/n} > n\Big(\frac{n+1}{2}\,m+p\Big)!\,.$$

Hence the requirements on f yield the claimed inequality. □

Using [2, item 3.6.55], i.e. the inequality

$$\Gamma(z)\geq z^{z-1/2}\,e^{-z}\sqrt{2\pi}$$

we get

Corollary 1.1′. With the same conditions as for Theorem 1′ it holds

$$f\big((m+p)!\big)+f\big((2m+p)!\big)+\cdots+f\big((mn+p)!\big) > n\cdot f\Big(n\cdot w^{w-1/2}\,e^{-w}\sqrt{2\pi}\Big) \tag{5}$$

where $w = w(m,n,p) = \dfrac{n+1}{2}\,m+p+1$.

Applications. By putting in (5)

$$p = 0 \text{ and } m \in (1,2) \text{ or}$$
$$p = -1 \text{ and } m = 2$$

we get more general inequalities than the ones given in [1, Corollary 1.1]. So, for instance $p = 0$ and $m = 2$ yield via $f(x) \equiv x$:

$$2!+4!+\cdots+(2n)! > n^2\left(\frac{n+2}{e}\right)^{n+2}\sqrt{\frac{2\pi}{n+2}}\,. \tag{6}$$

Remark. Similar to Theorem 1′ there also holds the following

Theorem 1″. Let $f\colon [1,\infty) \to R$ be a concave and decreasing function. Then

$$f((m+p)!)+f((2m+p)!)+\cdots+f((mn+p)!)<n\,f\Big(n\left(\frac{n+1}{2}\,m+p\right)!\Big)\,. \quad (7)$$

As an improvement of [1, Theorem 2] we state

Theorem 2′. With the same conditions as for Theorem 1′ it holds

$$(m+p)!^{(m+p)!}\,(2m+p)!^{(2m+p)!}\cdot\ldots\cdot(mn+p)!^{(mn+p)!}>g\Big(n\left(\frac{n+1}{2}\,m+p\right)!\Big)^{n} \quad (8)$$

where $g(x)\equiv x^x$

Proof. This result follows from Theorem 1′ and its Corollary upon putting $f(x) = x \ln x$. (Because of $f'(x) = 1+\ln x$ and $f''(x) = \dfrac{1}{x}$ the requirements of Theorem 1′ are satisfied.)

Theorem 2′ enables us to improve Corollary 1.2 of [1].

Thus for instance

$$2!^{2!}\cdot 4!^{4!}\cdot\ldots\cdot(2n)!^{(2n)!}>\left[n\,(n+1)!\right]^{n^2(n+1)!}\,. \quad (9)$$

Finally we prove as a compagnion of [1, Theorem 3] the following

Theorem 3′. For $n\ge 2$ it holds

$$\prod_{k=1}^{n}\left[k!\binom{n}{k}\right]^{k/n^k}\le\left[\frac{n^n(n-1)^2}{n^n-n}\right]^{(n-n^{2-n})/(n-1)^2}\,. \quad (10)$$

Proof. Starting from

$$\frac{1}{n}\cdot 1!\binom{n}{1}+\frac{2}{n^2}\cdot 2!\cdot\binom{n}{2}+\cdots+\frac{n}{n^n}\cdot n!\cdot\binom{n}{n}=n$$

we get via the weighted geometric – arithmetic means inequality and using

$$\frac{1}{n}+\frac{2}{n^2}+\cdots+\frac{n}{n^n}=\frac{n-n^{2-n}}{(n-1)^2}=:s(n)\,.$$

LHS(10) $\le [n/s(n)]^{s(n)}$ = RHS(10), and we are done. □

As a final remark we note $\lim_{n\to\infty}[n/s(n)]^{s(n)}=1$.

References

[1] Starc, Z. F.: *On Some Inequalities with Factorials.* Math. Moravica **1**, 101-194 (1997).

[2] Mitrinović, D. S.: *Analytic Inequalities*, Springer, Berlin (1970).

[3] Pečarić, J.: *On some Inequalities for Convex Functions and some Related Applications*, Mat. Bilten (Skopje), **5-6**, 29-36 (1981/1982).

НЕКОИ НЕРАВЕНСТВА СО ФАКТОРИЕЛИ

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Резиме

Во оваа работа се докажани некои неравенства за конечни суми и производи со факториели.

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