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A NOTE ON A LEMMA

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Abstract

In this work we give generalization of a lemma from [1] page 162. The Lemma is ilustrated by an example.

This note refers to lemma A.3.2 given in (1. p. 162) that is in context to the Fourier transform for the function $f \in L^1$. L^1 is the space of the integrable functions by Lebesgue in R.

Let the function $f \in L^1$, then the function

$$\hat{f}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z}\, dt, \quad z = x + iy$$

is analytic for $y \neq 0$. In relation to the mentioned lemma we take into consideration the function

$$f^*(x + iy) = \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{|t - z|^2}\, dt.$$

If $f(t)$ is a real-value function, then

$$f^*(x + iy) = \hat{f}(z) + \overline{\hat{f}}(z) = 2Re\hat{f}(z),$$

meaning that $f^*(x + iy)$ is a harmonic function for $y \neq 0$. From here, the function $f^*(x + iy)$ is called harmonic representation for the function f. It is known that $f^*(x+iy) \to f(x)$, when $y \to 0^+$, in the points $x \in R$ in which $f(x)$ is continuous, the convergence is uniform to compact sets. (1. 5.3 p, 69).

It is also of interest to mention that $f^*(x+iy) \to f(x)$ when $y \to 0^+$ in terms of distributions.

The lemma which our note refers to states:

Lemma 1. *Let $f_1, f_2 \in L^1$, and f_1^*, f_2^* are harmonic representations for the functions f_1, f_2. If it is true that $f_1^*(x+iy) = f_2^*(x+iy)$ for every $y \neq 0$, then $f_1(t) = f_2(t)$ almost everywhere.*

The lemma is equivalent to the condition: if $f \in L^1$ and $f^*(x+iy) = 0$ for $y \neq 0$, then $f(t) = 0$ almost everywhere.

The proof is given in (1. p. 162). Our note regarding the mentioned lemma is the following lemma:

Lemma 2. *Let $f_1, f_2 \in L^1$, and f_1^*, f_2^* are harmonic representations for the functions f_1, f_2. If it is true that $f_1^*(x+iy) = f_2^*(x+iy)$ for $y > 0$, then $f_1(t) = f_2(t)$ almost everywhere.*

Proof. We shall prove the lemma with the help of the analytic representation of the distributions. Each function $f \in L^1$ is a regular Schwartz distribution, where it has Cauchy representation. Namely, the function

$$\hat{f}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt, \quad z = x+iy$$

is analytic for $y \neq 0$ and in that $\hat{f}(x+iy) - \hat{f}(x-iy) \to f(x)$ when $y \to 0^+$ in terms of distributions (3. p. 17-19).

$$\hat{f}(x+iy) - \hat{f}(x-y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{|t-z|^2}\, dt = f^*(x+iy), \quad y > 0.$$

By analogy, we have $f^*(x+iy) \to f(x)$ when $y \to 0^+$ in terms of distributions. Supposedly, in the lemma $f_1^*(x+iy) = f_2^*(x+iy)$ for $y > 0$, we have

$$\lim_{y \to 0+} \int_{-\infty}^{\infty} f_1^*(x+iy)\varphi(x)dx = \lim_{y \to 0+} \int_{-\infty}^{\infty} f_2^*(x+iy)\varphi(x)dx$$

for every $\varphi \in D$. D is the space of the test functions in R. On the other hand we have

$$\lim_{y \to 0+} \int_{-\infty}^{\infty} f_1^*(x+iy)\varphi(x)dx \int_{-\infty}^{\infty} f_1(x)\varphi(x)dx$$

and

$$\lim_{y\to 0+} \int_{-\infty}^{\infty} f_2^*(x+iy)\varphi(x)dx = \int_{-\infty}^{\infty} f_2(x)\varphi(x)dx$$

from where it follows that $f_1(x) = f_2(x)$ almost everywhere (2. p. 66).

To illustrate this, we list one very important theorem from the Fourier transformation.

Example. If $f \in L^1$ is a continuous function and its Fourier transformation

$$F(f,x) = \int_{-\infty}^{\infty} f(t)e^{itx}\,dt$$

is also from L^1, then it is known that $f(t) = F^{-1}(F(f),t)$ for every t.

$$F^{-1}(g,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{ixt}dx, \quad g \in L^1$$

is called inverse Fourier transformation for the function g.

Let us now have $f \in L^1$. The function f will be not continuous, and let $F(f) \in L^1$. With the help of lemma 2., we will prove that $f(x) = F^{-1}(F(f),x)$ for almost every x, which is a known result.

For that purpose, we will first prove that, if $h(x) = f^*(x+i\varepsilon)$, $f \in L^1$, $\varepsilon > 0$ then

$$h^*(x+i\delta) = f^*(x+i(\delta+\varepsilon)), \quad \delta > 0.$$

$$\begin{aligned} h^*(x+i\delta) &= \frac{\delta}{\pi}\int_{-\infty}^{\infty} \frac{f^*(t+i\varepsilon)}{|t-x-i\delta|^2}\,dt \\ &= \frac{\delta}{\pi}\int_{-\infty}^{\infty} \frac{dt}{|t-x-i\delta|^2}\cdot\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{|\tau-t-i\varepsilon|^2}d\tau \end{aligned} \tag{1}$$

it is easily proved that

$$F(e^{i\omega x-\varepsilon|\omega|},t) = \int_{-\infty}^{\infty} e^{-i\omega x-\varepsilon|\omega|}e^{i\omega t}d\omega = \frac{2\varepsilon}{|t-x-i\varepsilon|^2}. \tag{2}$$

By setting in the relation (1) we get

$$h^*(x+i\delta) = \frac{\delta}{\pi}\int_{-\infty}^{\infty} \frac{dt}{|t-x-i\delta|^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)F(e^{-i\omega t-\varepsilon|\omega|},\tau)d\tau.$$

From here, with the help of the Parseval relation we have

$$h^*(x+i\delta) = \frac{\delta}{\pi}\int_{-\infty}^{\infty}\frac{dt}{|t-x-i\delta|^2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}F(f,\omega)e^{-i\omega t-\varepsilon|\omega|}d\omega$$

with change in the order of integration

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(f,\omega)e^{-\varepsilon|\omega|}\,d\omega\,\frac{\delta}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{|t-x-i\delta|^2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(f,\omega)e^{i(x-\delta|\tau|},t)e^{-i\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(f,\omega)e^{-\varepsilon|\omega|}e^{-i\omega x-\delta|\omega|}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(f,\omega)e^{-i\omega x}\cdot e^{-i\tau x-\delta|\tau|},t)e^{-i\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)F(e^{-i\omega x-(\varepsilon+\delta)\,|\omega|},t)\,dt\,.$$

From (2) we get

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\cdot\frac{2(\varepsilon+\delta)}{|t-x-i(\varepsilon+\delta)|^2}\,dt$$

$$= \frac{\varepsilon+\delta}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{|t-x-i(\varepsilon+\delta)|^2\pi}\,dt$$

$$= f^*(x+i(\varepsilon+\delta)).$$

The function $h(x) = F^{-1}(F(f),x)$ is continuous because it is an inverse

Fourier transforme. With the help of the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(f,\omega)e^{-i\omega x-\varepsilon|\omega|}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, F\left(e^{-i\omega t-\varepsilon|\omega|}, t\right) dt$$

$$= \int_{-\infty}^{\infty} f(t)\frac{\varepsilon}{\pi|t-x-i\varepsilon|^2}\, dt$$

$$= f^*(x+i\varepsilon)$$

we get

$$h(x) = \lim_{\varepsilon\to 0} f^*(x+i\varepsilon),$$

$$h^*(u+i\delta) = \frac{\delta}{\pi}\int_{-\infty}^{\infty} \lim_{\varepsilon\to 0} f^*(x+i\varepsilon)\frac{dx}{|x-u-i\delta|^2}$$

$$= \lim_{\varepsilon\to 0}\frac{\delta}{\pi}\int_{-\infty}^{\infty} f^*(x+i\varepsilon)\frac{dx}{|x-u-i\delta|^2}$$

using the result we got previously

$$h^*(u+i\delta) = \lim_{\varepsilon\to 0} f^*(u+i(\delta+\varepsilon)) = f^*(u+i\delta),$$

from where on the basis of lemma 2. $h(u) = f(u)$ almost everywhere. Since $h(u) = F^{-1}(F(f),u)$, we finally have $f(u) = F^{-1}(F(f),u)$ for almost every u.

References

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[2] Jantseher H.: *Distributionen,* Walter de Gryter, Berlin 1971.

[3] N. Rechkovski: *Une fonction analitique definie par une distribution,* God. Zbor. Mat. Faks., 31, 1980.

ЗАБЕЛЕШКА ЗА ЕДНА ЛЕМА

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Р е з и м е

Во оваа работа обопштена е една лема дадена во (1. стр. 162) лемата е илустрирана со еден пример.

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