Математички Билтен
16 (XLII)
1992 (59-61)
Скопје, Македонија

AN EXAMPLE OF AN INNER FUNCTION IN THE BALL

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Abstract. The existence of the inner functions in the unit ball B of $\mathbb{C}^n$ was proved by Aleksandrov [1] and Low [2]. Here we construct a sequence of holomorphic functions in the ball which converges to an inner function in the ball. This is also a new and simple proof of the existence of the inner functions.

Throughout this paper $\bar{a}z$ is the scalar product in $\mathbb{C}^n$, σ is the rotation Lebesgue measure on S and $Q(\zeta,r)$ are balls on S in the metric $d(\zeta,\eta)=|1-\bar{\zeta}\eta|^{1/2}$, $\zeta,\eta\in S$.

Cover S with disjoint family of balls $Q(\xi_k,r_k)=Q_k$, $k=1,2,\dots,m$. Let $\{\zeta_m\}$, $m\in N$, be a (dense) subset of S. Let λ_k, $k=1,2,\dots,m$ be complex numbers from the unit disc in the plane which satisfies

$$(\sigma Q_k)^2 e^{\{-(\sigma Q_k)^{-2}+(\sigma Q_k)^{-2}[1-(\sigma Q_k)^4]^{(\sigma Q_k)^{-2}}+2(\sigma Q_k)^{-1}\}} \le \tag{1}$$
$$\le 1-|\lambda_k|^2 \le (\sigma Q_k)^2\left[1-e^{-2(\sigma Q_k)^{-1}}\right],\quad k=1,2,\dots,m$$

Define the functions

$$u_m(z) = \prod_{k=1}^{m} \frac{\lambda_k-\bar{\zeta}_m z}{1-\bar{\lambda}_k\bar{\zeta}_m z}\,\frac{|\lambda_k|}{\lambda_k},\quad m\in N \tag{2}$$

Then $u_m(z)$, $m\in N$ are holomorphic functions in B and $|u_m(z)| < 1$ for every $z\in B$.

Theorem. The sequence $\{u_m(z)\}$ converge to a nonconstant inner function in the ball.

Proof. Let

$$\phi_{km}(z = \frac{\lambda_k-\bar{\zeta}_m z}{1-\bar{\lambda}_k\bar{\zeta}_m z}\,\frac{|\lambda_k|}{\lambda_k} \tag{1}$$

and

$$A_{km} = \frac{1-|\bar{\zeta}_m z|^2}{|1-\bar{\lambda}_k\bar{\zeta}_m z|^2} \tag{2}$$

Since

$$1-|\phi_{km}(z)|^2 = (1-|\lambda_k|^2)A_{km} \tag{3}$$

$$\frac{1-|z|}{1+|z|} \le A_{km} \le \frac{1+|z|}{1-|z|} \tag{4}$$

and $e^x \ge 1+x$, we have

$$\begin{aligned} 1-|\phi_{km}(z)|^2 &\ge \frac{1-|z|}{1+|z|}\left[-1+(1-\sigma Q_k^4)^{1/\sigma Q_k^2}+2\sigma Q_k\right] \ge \\ &\ge 2\frac{1-|z|}{1+|z|}\sigma Q_k-(\sigma Q_k)^2 \end{aligned} \tag{5}$$

which implies

$$\begin{aligned} |\phi_{km}(z)|^2 &\le 1-2\frac{1-|z|}{1+|z|}\sigma Q_k+(\sigma Q_k)^2 \le \\ &\le e^{-2\frac{1-|z|}{1+|z|}\sigma Q_k+o(\sigma Q_k)} \end{aligned} \tag{6}$$

and so

$$|u_m(z)|^2 \le e^{-\sum_{k=1}^{m} 2\frac{1-|z|}{1+|z|}\sigma Q_k+o(\sigma Q_k)} \tag{7}$$

Letting $m \to \infty$ we have

$$|u(z)| \ge e^{-\frac{1-|z|}{1+|z|}} \tag{8}$$

Using the right side in (1) we have

$$\begin{aligned} 1-|\phi_{km}(z)|^2 &\le \frac{1+|z|}{1-|z|}(\sigma Q_k)^2 - \frac{1-|z|}{1+|z|}e^{-2(\sigma Q_n)^{-1}}(\sigma Q_n)^2 = \\ &= \frac{1+|z|}{1-|z|}(\sigma Q_k)^2 - \frac{1-|z|}{1+|z|}(\sigma Q_k)^2+2\frac{1-|z|}{1+|z|}\sigma Q_k \end{aligned} \tag{9}$$

and so

$$|u_m(z)|^2 \ge e^{-\sum_{k=1}^{m} 2\frac{1-|z|}{1+|z|}\sigma Q_k+o(\sigma Q_k)} \tag{10}$$

Then letting $m \to \infty$, (10) implies

$$|u(z)| \ge e^{-\frac{1-|z|}{1+|z|}} \tag{11}$$

and with (7) we get

$$|u(z)| = e^{-\frac{1-|z|}{1+|z|}} \tag{12}$$

So, $u(o) = e^{-1}$ and

$$|u^*(\zeta)| = 1 \quad \text{a.e. } \sigma \text{ on } S.$$

R E F E R E N C E S

[1] Aleksandrov, B.: The Existence of the inner functions in the ball, Math. Sb. 118(160), (1982), N^{o} 2(6), 147-163

[2] Low, E.: A construction of inner functions on the unit ball in C^{p}, Inv. Math. 67(1982), 223-229

ПРИМЕР НА ВНАТРЕШНА ФУНКЦИЈА ВО ТОПКАТА

Н. Пандески

Р е з и м е

Во работава е конструирана низа холоморфни функции во топката која конвергира кон неконстантна внатрешна функција. Со тоа е даден нов едноставен доказ на постоењето на внатрешните функции во топката како и експлицитен пример на внатрешна функција.