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FOR TWO THEOREMS OF THE ANALYTIC REPRESENTATION OF DISTRIBUTIONS

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In this article is given the proof for two theorems of the analytic representation of distributions

The two theorems that we consider here conserns to the analytic representation of distributions. Namely, it is known that for every distribution $T \in D'$, where D' is the spaces of the Schwarz distributions on the real line R, there is a complex function $f(z), z = x + iy$ that is analytic for $y \neq 0$ and in that

$$\lim_{\varepsilon \to 0} [\int_{-\infty}^{\infty} (x + i\varepsilon) - f(x - i\varepsilon)] \varphi(x) dx = \langle T, \varphi \rangle = T(\varphi) \quad \varphi \in D \qquad (1)$$

D is the space of test function on R.

The function $f(z)$ is called analytic representation of the distribution T. Actually, the function $f(z)$ is analytic on the complex plane C, exept on the support "supp T" of the distribution, T see (1.p.76).

To compute the analytic representation for a given distribution T, in general case, is not easy. However, if the supp T ia a complete set, then the function

$$\hat{T}(z) = \frac{1}{2\pi i} \langle T_t, \frac{1}{t - z} \rangle, y \neq 0 \qquad (2)$$

is a analytic representation for the distribution T (1.p.73).

Other terminologies used for $\hat{T}(z)$ are the Cauchy representation of the analytic representation of T by means of the Cauchy cernel.

In order to represent as many distributions as possible with the Cauchy integral, H. Bremermann (1) introduced the test spaces $O_\alpha = O_\alpha(R^n)$ and distributions spaces O'_α.

Definition. *For α being a given real number, we say that a function $\varphi \in O_\alpha$, if φ is infinitely differentiable and it for each n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of nonnegative integers there exists a constant A, such that*

$$|D^\gamma \varphi(x)| \leq A(1+|x|)^\alpha, \quad x \in R.$$

Convergence in the vector space O_α is defined as follows: The sequence (φ_j) converges to zero in O_α if for each γ the sequence $(D^\gamma \varphi_j)$ converges uniformly on every compact subset of R^n to zero as $j \to \infty$, and for each γ there exists constant A which is independent of j, such that

$$|D^\gamma \varphi_j(x)| \leq A(1+|x|)^n, \quad x \in R^n.$$

Definition. *For a distribution $T \in D'$ we write $T = O(|t|^\beta)$, β is a real number, if there are constants R asnd A such that for all $\varphi \in D$ with support in $\{x \in R^n \colon |x| > R\}$ we have*

$$|\langle T, \varphi\rangle| \leq A \int\limits_{R^n} |x|^\beta \, |\varphi(x)| dx\,.$$

For such distribution is valid the following theorem.

Theorem 1. *Let $T \in D'$ and $T = (|t|^\beta)$. Then T can be extendet to O'_α for any α such that $\alpha + \beta + n < 0$, where n is the dimension R^n; further, the extension is unique.*

For complete proof see (1.p.85).

Here we give proof for the theorem 1.

Proof. Let as consider the sequence $\{\alpha_N(x)\}$, $\alpha_N \in D$, $0 \leq \alpha_N \leq 1$ and $\alpha_N(x) = 1$ for $|x| \leq N$.

Let the function $\varphi \in O_\alpha$, α is a real number with $\alpha + \beta + n < 0$. Now we consider the sequence $\{\langle T, \alpha_N \varphi\rangle\}$. $(*)$

Clearly the function $\alpha_N \varphi \in D$. We choose $N_\circ$ such that $|\varphi(x)| \leq A|x|^\alpha$ for $|x| \geq N_\circ, N_\circ > R$. We shall prove that $(*)$ is a Cauchy sequence. For $N_1 > N_1 \geq N_\circ$ we have

$$|\langle T, \alpha_{N_2}\varphi\rangle - \langle T, \alpha_{N_1}\varphi\rangle| = |\langle T, (\alpha_{N_2} - \alpha_{N_1}\varphi\rangle|$$

$\operatorname{supp}(\alpha_{N_2} - \alpha_{N_1})\varphi \subset \{x \colon |x| \geq N_\circ\}$. Consequently on obtain $|\langle T, (\alpha_{N_2} - \alpha_{N_1})\varphi| \leq \int\limits_{|x| \geq N_\circ} |x|^{\alpha+\beta} dx$, for the constant $a = \max(A, A_1)$.

By introducing the polar coordinates we have $\omega_n \int\limits_{N_\circ}^{\infty} \rho^{\alpha+\beta+n-1} d\rho$, ω_n is the area of the sphere with radius 1. Because

$$\int_{N_\circ}^{\infty} \rho^{\alpha+\beta+n-1} d\rho = \frac{1}{\alpha+\beta+n} N_\circ^{\alpha+\beta+n} \to 0$$

as $N_\circ \to \infty$ the proof is complete. Thus the limes

$$\Lambda(\phi) = \lim_{N\to\infty} \langle T, \alpha_N \varphi \rangle \quad \text{exists for all} \quad \varphi \in O_\alpha . \tag{3}$$

The linearity of the functional is evident.

Now we must to prove that the functional is continuous O_α.

Let $\{\varphi_j\}$ be any sequence which converges to zero in O_α as $j \to \infty$. We shall to prove that $\Lambda(\varphi_J) \to 0$ as $j \to \infty$. Let $R > 0$ be so that $|\varphi_j(x)| \leq A|x|^\alpha$ for all φ_j, if $|x| > R$. Then

$$|\Lambda(\varphi_j)| = \lim_{N\to\infty} \langle T, \alpha_N \varphi_j \rangle| = \lim_{N\to\infty} \{\, |\langle T, (\alpha_N - \alpha_{N_\circ})\varphi_j \rangle + \langle T, \alpha_{N_\circ}\varphi_J \rangle| \,\}$$
$$\leq \lim_{N\to\infty} \{\, |\langle T, (\alpha_N - \alpha_{N_\circ})\varphi_j \rangle| + |\langle T, \alpha_{N_\circ}\varphi_j \rangle| \,\} .$$

If $N \geq N_\circ$ and $N_0 > R$, $\operatorname{supp}(\alpha_N - \alpha_{N_\circ})\varphi_j \subset \{x{:}\, |x| \geq N_0 > R\}$ then $|\langle T, (\alpha_N - \alpha_{N_\circ})\varphi_j \rangle| \leq \int\limits_{|x|>N_\circ} |x|^{\alpha+\beta} dx$ if $N_\circ$ sufficiently large. Let $\varepsilon > 0$ is given, we can choose $N_\circ$ so that, $|\langle T, (\alpha_N - \alpha_{N_\circ})\varphi_j \rangle| < \varepsilon$ for every $N \geq N_\circ$. Further, the sequence $\{\alpha_{N_\circ}\varphi_j\}$ converges to zero in D as $j \to \infty$. Thus we have

$$|\langle T, \alpha_{N_\circ}\varphi_j \rangle| < \varepsilon \quad \text{if} \quad j \geq j_\circ .$$

Combining these facts on obtain $|\Lambda(\varphi_j)| < 2\varepsilon$ if $j \geq j_\circ$. Thus $\lim\limits_{j\to\infty} \Lambda(\varphi_j) = 0$. Let now the function $\varphi \in D$ then $\operatorname{supp}\varphi \subset \{x{:}\, |x| < N_\circ\}$ for sufficiently large $N_\circ$ so that $\alpha_N \varphi = \varphi$ if $N > N_\circ$. Consequently we have

$$\Lambda(\varphi) = \lim_{N\to\infty} \langle T, \alpha_N \varphi \rangle = \lim_{N\to\infty} \langle T, \varphi \rangle = T(\varphi) \quad \text{for} \quad \varphi \in D .$$

Thus Λ is an extension for T. Since D is dense in O_α we obtain that the functional is unique.

The other theorem that we will consider here is given in (2, p.46) as a lemma.

Lemma. (2, p.46). *Let $h^+(z)$ be analytic in $\Lambda^+ = \{z{:}\, y > 0\}$ with $h^+(z) = O\left(\frac{1}{|z|}\right)$ in Λ^+; Let $h^+(x + i\varepsilon)$ converge to the boundary value $h^+ \in D(R)$ as $\varepsilon \to 0^+$; that is let $\langle h^+, \varphi \rangle = \lim\limits_{\varepsilon\to 0_+} \int\limits_{-\infty}^{\infty} h^+(x + i\varepsilon)\varphi(x)dx$ for all $\varphi \in D$. Then*

(i) $h^+ \in O_\alpha(R)$ *for all* $\alpha < 0$ *and*
(ii) $h^+(x+i\varepsilon)$ *converges to* h^+ *in the* O_α *topology,* $\alpha < 0$ *as* $\varepsilon \to 0_+$.
Additionaly, if $-1 \le \alpha < 0$ *we have*

(iii) $$\frac{1}{2\pi i} < h_t^+, \frac{1}{t-z}\rangle = \begin{cases} h^+(z), & z \in \Lambda^+ \\ 0, & z \in \Lambda^- \end{cases} \qquad \Lambda^- = \{z : y < 0\}.$$

Here, we first made some remarks to the proof of part (iii) *and then we give another proof for* (ii) *formally stated as a theorem.*

Remarks: (a) On the page 49 state

$$\lim_{\varepsilon\to 0} 0_+ \int\limits_{|x|\le r} h^+(x+i\varepsilon)\varphi(x)dx = \langle h^+, \varphi\rangle \quad \varphi \in O_\alpha.$$

This limes is not in accordance with the hypothesis in lemma, because in the lemma state

$$\lim_{\varepsilon\to 0_+} \int\limits_{-\infty}^{\infty} h^+(x+i\varepsilon)\varphi(x)dx = \langle h^+, \varphi\rangle \quad \text{only for} \quad \varphi \in D$$

(b) p.49 It is not clear, why $h'(x+i\varepsilon) \to H^+(x) \in L^\infty$ as $\varepsilon \to 0_+$? Further "using the Lebesque dominated convergence theorem we obtain"

$$\lim_{\varepsilon\to 0_+} \int\limits_{|x|>r} h^+(x+i\varepsilon)\varphi(x)dx = \int\limits_{|x|>r} H^+(x)\varphi(x)dx?$$

Remark. if the function $\varphi \in O_\alpha$, $-1 < \alpha < 0$ in general $H^+(x)\varphi(x)$ is not integrable.

(c) Combining these fact, there exists an element $U \in O'_\alpha$ $\alpha < 0$ such that

$$\lim_{\varepsilon\to 0}\langle h^+(x+i\varepsilon), \varphi(x)\rangle = U(\varphi), \quad \varphi \in O\alpha.$$

"This implies ... $h^+ = U$ on O_n, $\alpha < 0$"?

Remark. 1) $\lim_{\varepsilon\to 0_+} \int\limits_{|x|\le r} h^+(x+i\varepsilon)\varphi(x)dx = \langle h^+, \varphi\rangle, \quad \varphi \in O_\alpha$

2) $\lim_{\varepsilon\to 0_+} \int\limits_{|x|>r} h^+(x+i\varepsilon)\varphi(x)dx = \int\limits_{|x|>r} H^+\varphi(x)dx.$

Thus $\langle h^+, \varphi\rangle + \int\limits_{|x|>r} H^+\varphi(x)dx = \langle h^+, \varphi\rangle$?

In following we give another proof for part (ii) as theorem?.

Theorem 2. *Let* $h^+(z)$ *be analytic in* Λ^+ *with* $h^+(z) = O\left(\frac{1}{|z|}\right)$. *Let* $h^+(x+i\varepsilon)$ *converge to the distribution* $h^+ \in D'$ *as* $\varepsilon \to 0_+$, *that is means*

$$\langle h^+, \varphi\rangle = \lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} h^+(x+i\varepsilon)\varphi(x)dx \quad \text{for all} \quad \varphi \in D\,. \tag{4}$$

Then $h^+(x+i\varepsilon)$, h^+ are in O'_α, $\alpha < 0$ and tehe relation (4) hold for all $\varphi \in O_\alpha$, $\alpha < 0$.

Proof. By using the asymptotic bounds and the and the theorems on obtain immediately that the distributions $h^+(x+i\varepsilon)$, h^+ can be extended from D to O'_α, $\alpha < 0$.

We now prove that $h^+(x+i\varepsilon) \to h^+$ in O'_α, $\alpha < 0$.

Let $\varphi \in O_\alpha$. We shall prove that the $\lim\limits_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} h^+(x+i\varepsilon)\varphi(x)dx$ exist.

For this purpose we consider the integrals $\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\varphi(x)dx$. From the asimptotic bound for φ and $h^+(z)$ we can choose constants R and A so that

$$\Big|\int_{|x|>R} h^+(x+i\varepsilon)\varphi(x)dx\Big| \le A \int_{|x|>R} \frac{dx}{|x|^{1-\alpha}}\,, \qquad \alpha < 0\,. \tag{5}$$

Let $\alpha(x) \in D$ be such that $0 \le \alpha(x) \le 1$, $\alpha(x) = 1$ for $x \in [-N, N]$ and $N > R$ is sufficiently large so that

$$A \int_{|x|>N} \frac{dx}{|x|^{1-\alpha}}\,, \qquad \text{for} \quad \beta > 0\,.$$

Further

$$\langle h^+(x+i\varepsilon), \varphi(x)\rangle = \langle h^+(x+i\varepsilon), \alpha(x)\varphi(x)\rangle + h^+(x+i\varepsilon), \big(1-\alpha(x)\big)\varphi(x)\rangle$$

(a) $\Big|\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\big(1-\alpha(x)\big)\varphi(x)dx\Big| \le A \int_{|x|>N} \frac{dx}{|x|^{1-\alpha}} < \beta$ from (5).

(b) $\langle h^+(x+i\varepsilon), \alpha(x)\varphi(x)\rangle \to \langle h^+, \alpha\varphi\rangle$ as $\varepsilon \to 0^+$ $\alpha(x)\varphi(x) \in D$ that is for $0 < \varepsilon \le \varepsilon_0$ we have $|\langle h^+(x+i\varepsilon), \alpha(x)\varphi(x)\rangle - \langle h^+, \alpha\varphi\rangle| < \dfrac{\delta}{2}$, consequently

$$\begin{aligned}&|\langle h^+(x+i\varepsilon_1), \alpha(x)\varphi(x)\rangle - \langle h^+(x+i\varepsilon_2), \alpha(x)\varphi(x) > | \le\\ &\le |\langle h^+(x+i\varepsilon_1), \alpha\varphi\rangle - \langle h^+, \alpha\varphi\rangle| + |\langle h^+(x+i\varepsilon_2), \alpha\varphi\rangle - \langle h^+, \alpha\varphi\rangle| < \delta\end{aligned}$$

this means that

$$\begin{aligned}&|\langle h^+(x+i\varepsilon_1), \varphi(x)\rangle - \langle h^+(x+i\varepsilon_2), \varphi(x)\rangle| \le\\ &\le |\langle h^+(x+i\varepsilon_1), \alpha\varphi\rangle - \langle h^+(x+i\varepsilon_2), \alpha\varphi\rangle| + |\langle h^+(x+i\varepsilon_1), (1-\alpha)\varphi\rangle|\\ &+ |\langle h^+(x+i\varepsilon_2), (1-\alpha)\varphi\rangle| < 3\delta\end{aligned}$$

if $0<\varepsilon_1,\varepsilon_2<\varepsilon_0$. Thus it exists

$$U(\varphi)=\lim_{\varepsilon\to 0_+}\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\varphi(x)dx \quad \text{for all} \quad \varphi\in O_\alpha, \quad \alpha<0\,. \tag{6}$$

Evidently the functional U is linear. To complete the proof we must show that U is continuous. Let $\{\varphi_j\}$ be any sequence of the space O_α that converges to zero, this means that the sequence converges uniformly on the compact subset and also it exists the constants a, R so that $\varphi_j(x)|\le \alpha|x|^\alpha$ for $|x|>R$, indepedently of j. By taking into account the asymptotic bound of the function $h^+(z)$ we can choose a function $\alpha(x)\in D$ $0\le\alpha(x)\le 1$, $\alpha(x)=1$ for $|x|\le N$ amd $N>R$ so large that

$$A\frac{dx}{|x|^{1-\alpha}}<\delta \quad \text{for a given} \quad \delta>0 \tag{$*$}$$

$$\begin{aligned}&|U(\varphi_j)|=\\ &=\lim_{\varepsilon\to 0_+}\left|\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\alpha(x)\varphi_j(x)dx+\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\big(1-\alpha)(x)\big)\varphi_j(x)dx\right|\\ &\le\lim_{\varepsilon\to 0_+}\left\{\left|\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\alpha(x)\varphi_j(x)dx\right|+\left|\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\big(1-\alpha(x)\big)\varphi_j(x)dx\right|\right\}.\end{aligned}$$

Because supp $\big(1-\alpha(x)\big)\varphi_j(x)\subset\{x\colon |x|>N\}$ for all j from $(*)$ we have

$$\Big|\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\big(1-\alpha(x)\big)\varphi_j(x)dx\Big|<\delta\,.$$

Thus on obtain

$$|U(\varphi_j)|\le\left|\lim_{\varepsilon\to 0_+}\int_{-\infty}^{\infty} h^+(x+i\varepsilon)\alpha(x)\varphi_j(x)dx\right|+\delta=|\langle h^+,\alpha\varphi_j\rangle|+\delta\,.$$

The sequence $\{\alpha\varphi_j\}$ converges to zero in D and from the hypothesis in the theorem $\langle h^+,\alpha\varphi_j\rangle\to 0$ as $j\to\infty$.

Consequently for $j\ge j_0|\langle h^+\alpha\varphi_j\rangle|<\delta$. Finally we have $|U(\varphi_j)|<2\delta$ for $j\ge j_0$ or $\lim\limits_{j\to\infty}U(\varphi_j)=0$. Thus $U\in O'_\alpha$, $\alpha<0$.

Let the function $\varphi \in D$, then $\operatorname{supp}\varphi \subset (-N, N)$ for sufficiently large N. In that case

$$U(\varphi) =$$

$$= \lim_{\varepsilon \to 0_+} \left\{ \int_{-\infty}^{\infty} h^+(x+i\varepsilon)\alpha(x)\varphi(x)dx + \int_{-\infty}^{\infty} h^+(x+i\varepsilon)\big(1-\alpha(x)\big)\varphi(x)dx \right\}$$

$$= \lim_{\varepsilon \to 0_+} \int_{-\infty}^{\infty} h^+(x+i\varepsilon)\varphi(x)dx + 0 = h^+(\varphi)\,.$$

But $D(R)$ is dense in O_α hence $h^+ = U$ on O_α. With this the proof of theorem is the complete.

Example $h^+(z) = -\frac{1}{2\pi i z}$, $h^+(x+i\varepsilon) \to \delta^+ = \dfrac{1}{2}\delta - 1\dfrac{1}{2\pi i}ch\dfrac{1}{t}$.

Obviously, coresponding results hold for a given analytic function $h^-(z)$ in λ with $h^-(z) = O\Big(\dfrac{1}{|z|}\Big)$ and for which there exists $h^- \in D$ such that

$$\langle h^-, \varphi\rangle = \lim_{\varepsilon \to 0_+} \langle h^-(x-i\varepsilon), \varphi(x)\rangle$$

$$= \lim_{\varepsilon \to 0_+} \int_{-\infty}^{\infty} h^-(x-i\varepsilon)\varphi(x)dx \quad \text{for all} \quad \varphi \in D(R)\,.$$

Remark. In the lemma it is sufficient only the existence of the $\lim\limits_{\varepsilon \to 0_+} \langle h^+(x+i\varepsilon), \varphi(x)\rangle$ for all $\varphi = inD$, because every subspace $D_K(R)$, K is a compact subset, is a Frechet space and also a linear functional on D is continuous if and only if continuous on every subspace D_K.

References

[1] Bremerman G.: *Distribution complex variables, and Furier transforms,* Addison-Wesley, New York, 1965.

[2] E. J. Beltrami and M. R. Wohlers: *Distributions an the boundary values of analytic functions,* Academic Press, New York, 1966.

[3] R. Carmichael, D. Mitrović: *Distributions and analytic functions,* New York, 1989.

ЗА ДВЕ ТЕОРЕМИ ОД АНАЛИТИЧНАТА ПРЕЗЕНТАЦИЈА НА ДИСТРИБУЦИИ

Васко Речкоски

Р е з и м е

Во оваа работа се дадени нови докази за две теореми од аналитичната репрезентација на дистрибуциите. Тоа се Теорема 1 и Теорема 2. Даден е исто така и еден пример и една забелешка.

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