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ON ESTIMATION OF FOURIER AND QUASI-MONOTONE-FOURIER COEFFICIENTS OF FUNCTIONS IN NIKOL'SKII CLASSES

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Abstract. An equivalent form of Nikol'skii class $(H(p,k,\varphi), p \in [1,\infty])$ using a supplementary condition for function φ and best approximation is given. The estimation of Fourier coefficients of functions belonging to the class $H(p,k,\varphi), p \in [1,\infty]$, by means of best approximation and modulus of smoothness without giving supplementary conditions to the φ function is investigated. We discuss the problem for the functions with quasi-monotone Fourier coefficients as well.

1. Introduction

The main problem in the approximation theory is to determine the properties of the approximate function characteristics based on the axiomatic properties of the function, as its modulus of smoothness and the constructive characteristics of that function and as its best approximation by trigonometric polynomials and its Fourier coefficients, which is well-known relation, [3, 4, 9, 10].

In this paper we consider the problem of estimation of Fourier coefficients of functions belonging to Nikol'skii class further work of [1, 2] and especially quasi-monotone Fourier coefficients of functions belonging to Nikol'skii class as a subclass of L_p spaces. The estimation is based on best approximation and modulus of smoothness. Initially, we will give some concepts, definitions and notations.

We denote by L_p the set of 2π-periodic functions f, such that f is measurable on $[0,2\pi]$ for $p \in [1,\infty)$ and f is continuous on $[0,2\pi]$ for $p=\infty$

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and respectively

$$\|f\|_{L_p}=\begin{cases}\left(\int\limits_0^{2\pi}|f(x)|^p dx\right)^{1/p}<\infty,\ \text{for } p\in[1,\infty).\\ \max\limits_x|f(x)|\,,\text{for } p=\infty.\end{cases}$$

Denote by $\omega_k(f,t)_p$ the modulus of smoothness of order k of the function f belonging to metrics L_p and respectively

$$\omega_k(f,t)_p \sup_{h\in[-t,t]}\left\|\triangle_h^k f(x)\right\| = \sup_{h\in[-t,t]}\left\|\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}f(x+mh)\right\|.$$

Denote by $E_n(f)_p$ the best approximation of the function $f\in L_p$ by trigonometric polynomials with degree not greater than n, where $n\in\mathbb{N}$ and define it respectively by $E_n(f)_p=\inf\limits_{H_n}\|f-H_n(x)\|_p$, where $H_n(x)$ are trigonometric polynomials of degree n.

Definition 1.1. By $L(0,2\pi)$ we denote the set of 2π-periodic functions summable on $(0,2\pi)$. Let $f\in L(0,2\pi)$ and let $a_m=a_m(f)$ and $b_m=b_m(f)$, $m=0,1,2,\ldots$ be Fourier coefficients of f, respectively, and

$$S[f]=\frac{a_0}{2}+\sum_{m=1}^{\infty}(a_m\cos mx+b_m\sin mx).$$

Remark 1.1: For any $p,q,1<p<q<\infty$ it is clear that $C\subset L_\infty\subset L_q\subset L_p\subset L$. We will write the Fourier series of the function in it's complex form, respectively by

$$f(x)\backsim\sum_{m=-\infty}^{\infty}C_m e^{imx},$$

where $C_m=\dfrac{1}{2\pi}\int_0^{2\pi}f(u)e^{-imu}du$. By $S_n[f;x]$ we will denote partial sum of the Fourier series of the function f, respectively by

$$S_n[f;x]=\sum_{m=-n}^{n}C_m e^{imx}.$$

Definition 1.2. 2π-periodic function $f(x)$ belongs to the class $H(p,k,\varphi)$, $p\in[1,\infty]$ if $f(x)\in L_p$, $\omega_k(f,\theta)_p<A\varphi(\theta)$ and the function $\varphi(\theta)$ has the following properties:

i) $\varphi(\theta)$ is a non-negative and continuous function on $[0,1]$ and $\varphi(\theta)\neq 0$,
ii) $\varphi(\theta)\leq A_1\varphi(\theta_2)$ for $0\leq\theta_1\leq\theta_2\leq 1$,
iii) $\varphi(2\theta)\leq A_2\varphi(\theta)$ for $0\leq\theta\leq\dfrac{1}{2}$,
where A,A_1 and A_2 are constants not dependent on θ,θ_1 and θ_2.

Definition 1.3. ([8]) The sequence of positive numbers $\{a_n\}$, $n \in \mathbb{N}$ is called *quasi-monotone* if

$$\frac{a_n}{n^{-\tau}} \downarrow 0. \tag{1.1}$$

for any $\tau > 0$. The sequence of inequalities $(a_1 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots \geq 0)$ is denoted by $a_n \downarrow 0$.

2. Auxiliary Facts

In this section we present some additional results used in our main results.

Proposition 2.1. ([5, 9]) *Let $f(x) \in L_p$. Than the following inclusions are valid:*

$$\omega_k(f,t)_p \in \begin{cases} \left(0, \dfrac{A_1}{n^k}\sqrt[p]{\displaystyle\sum_{m=1}^{n} m^{mp-1}E_m^p(f)_p}\,\right], p \in (1,2] \\ \left[\dfrac{A_1}{n^k}\sqrt[p]{\displaystyle\sum_{m=1}^{n} m^{mp-1}E_m^p(f)_p}, \infty\right), p \in [2,\infty) \end{cases} \tag{2.1}$$

where A_1 and A_2 are constants not dependent on $f(x)$ and n.

Proposition 2.2. ([6]) *Let $f(x) \in L_p$ than the following inclusions are valid:*

i) For $p = 1$,

$$A_1|C_n| \leq \omega_k(f,t)_p \leq A_2\left\{\frac{\sqrt{\sum\limits_{|m|=1}^{n}|m|^{2k}|C_m|^2}}{n^k} + \sqrt{\sum\limits_{|m|=n+1}^{\infty}|C_m|^2}\right\}$$

ii) For $p \in (1,2]$,

$$A_1\left\{\frac{\sqrt[p]{\sum\limits_{|m|=1}^{n}|m|^{p(k+1)-2}|C_m|^p}}{n^k} + \sum\limits_{|m|=n+1}^{\infty}|m|^{p-2}|C_m|^p\right\} \leq$$

$$\leq \omega_k(f,t)_p \leq A_2\left\{\frac{\sqrt{\sum\limits_{|m|=1}^{n}|m|^{2k}|C_m|^2}}{n^k} + \sqrt{\sum\limits_{|m|=n+1}^{\infty}|C_m|^2}\right\}$$

iii) For $p \in [2, \infty)$,

$$A_1 \left\{ \frac{\sqrt{\sum\limits_{|m|=1}^{n} |m|^{2k} |C_m|^2}}{n^k} + \sqrt{\sum\limits_{|m|=n+1}^{\infty} |C_m|^2} \right\} \leq \omega_k(f,t)_p \leq$$

$$\leq A_2 \left\{ \frac{\sqrt[p]{\sum\limits_{|m|=1}^{n} |m|^{p(k+1)-2} |C_m|^p}}{n^k} + \sum\limits_{|m|=n+1}^{\infty} |m|^{p-2} |C_m|^p \right\}$$

iv) For $p = \infty$,

$$A_1 \left\{ \frac{\sqrt{\sum\limits_{|m|=1}^{n} |m|^{2k} |C_m|^2}}{n^k} + \sqrt{\sum\limits_{|m|=n+1}^{\infty} |C_m|^2} \right\} \leq$$

$$\leq \omega_k(f,t)_p \leq A_2 \left\{ \frac{\sqrt{\sum\limits_{|m|=1}^{n} |m|^{k} |C_m|}}{n^k} + \sqrt{\sum\limits_{|m|=n+1}^{\infty} |C_m|} \right\}$$

where A_1 *and* A_2 *are constants not dependent on* $f(x)$ *and* n.

Proposition 2.3. ([9]) *If* $f(x) \in L_p, p \in (1, \infty)$, *then for every natural number* n *the following inequality holds:*

$$\omega_k(f, \frac{1}{2^n})_p \leq 2^{-nk} A \left(E_0(f)_p + \sum_{m=0}^{n} 2^{mk} E_{2^m}(f)_p \right) \tag{2.2}$$

Proposition 2.4. ([7]) *Let* $\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx + b_m + \sin mx$ *be Fourier series of a function* $f(x) \in L_p$, $p \geq 1$. *Then*

$$\left| \sum_{m=n+1}^{2n} (a_m + b_m) \right| \leq A \cdot \sqrt[p]{n} E_n(f)_p \tag{2.3}$$

where A *is a constant not dependent on* $f(x)$ *and* n.

3. Main results

Theorem 1. *Let the function $\varphi(\theta)$ satisfy this supplementary condition:*

$$\int_{\theta}^{1} \frac{\varphi(v)}{v^{k+1}} dv \leq C \frac{\varphi(\theta)}{\theta^k}$$

The function $f(x) \backsim \sum_{m=-\infty}^{\infty} C_m e^{imx}$ belongs to the class $H(p,k,\varphi)$, $p \in [1,\infty]$ if and only if

$$E_n(f)_p \leq A(n+1)^k \frac{\varphi^2\left(\frac{1}{n+1}\right)}{\int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}} dv},$$

where constants C and A are not dependent on θ.

Proof. Necessity condition. According to the the Jackson's theorem, properties of the modulus of smoothness, the inequality $n+1 < 2n$ and the definition of the class $H(p,k,\varphi)$, $p \in [1,\infty]$ we have:

$$\begin{aligned} &E_n(f)_p \leq A_1 \omega_k \left(f, \tfrac{1}{n}\right)_p \leq A_1 \omega_k \left(f, \tfrac{2}{n+1}\right)_p \leq \\ &\leq A_2 \omega_k \left(f, \tfrac{1}{n+1}\right)_p \leq A_2 A_3 \varphi\left(\tfrac{1}{n+1}\right) = A_4 \varphi\left(\tfrac{1}{n+1}\right) \end{aligned}$$

where $A_2 \geq 2^k A_1$. From the other side, for $\theta = \left(\frac{1}{n+1}\right)$ we have

$$\varphi\left(\frac{1}{n+1}\right) \geq \frac{\int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}} dv}{C(n+1)^k}.$$

Applying arithmetic mean inequality we obtain

$$\varphi\left(\frac{1}{n+1}\right) \geq \frac{1}{2}\left(\frac{\int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}} dv}{C(n+1)^k} + \frac{E_{n(f)_p}}{A_4}\right) \geq \sqrt{\frac{E_{n(f)_p}}{A(n+1)^k} \int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}} dv}$$

and respectively, $E_n(f)_p \leq A(n+1)^k \dfrac{\varphi^2(\frac{1}{n+1})}{\int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}}dv}$, where $A = CA_4$.

Sufficiency condition. From the relations

$$E_n(f)_p \leq A(n+1)^k \frac{\varphi^2\left(\frac{1}{n+1}\right)}{\int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}}dv}$$

and

$$\varphi\left(\frac{1}{n+1}\right) \geq \frac{\int\limits_{\frac{1}{n+1}}^{1} \frac{\varphi(v)}{v^{k+1}}dv}{C(n+1)^k}$$

we have that $\frac{A}{C}\varphi\left(\frac{1}{n+1}\right) = A_5\varphi\left(\frac{1}{n+1}\right) \geq E_n(f)_p$. Since the function φ is continuous and uniformly bounded on the segment $[0,1]$, for a fixed $n \in \mathbb{N}$, we take the trigonometric polynomial of the best approximation. Since $H_nf \in L_\infty$, and respectively $H_nf \in L_p$, using the triangle inequality we obtain:

$$\|f\|_p = \|f + H_nf - H_nf\|_p \leq \|f - H_nf\|_p + \|H_nf\|_p = E_n(f)_p + \|H_nf\|_p \leq$$
$$\leq A_6\varphi\left(\frac{1}{n+1}\right) + \|H_nf\|_p \leq \max_{0\leq\theta\leq 1}\{A_6\varphi(\theta) + \|H_nf\|\} = A_7.$$

So, $f(x) \in L_p$. For this reason, since $f(x) \in L_p$ from the relation (2.2) and the properties of the function φ, the following relations hold:

$$\omega_k\left(f, \frac{1}{2^n}\right) \leq 2^{-nk}A_8\left(E_0(f)_p + \sum_{m=0}^{n} 2^{mk}E_{2^m}(f)_p\right) \leq$$
$$\leq 2^{-nk}A_8\left(A_9\left(\varphi(1) + \sum_{m=0}^{n} 2^{mk}\varphi\left(\frac{1}{2^m+1}\right)\right)\right) \leq$$
$$\leq 2^{-nk}A_{10}\left(\varphi(1) + \varphi\left(\frac{1}{2}\right) + \sum_{m=1}^{n} 2^{mk}\varphi\left(\frac{1}{2^m+1}\right)\right) \leq$$
$$\leq 2^{-nk}A_{10}\left(\varphi(1) + A_{11}\varphi(1) + (1+A_{11})\sum_{m=1}^{n} 2^{mk}\varphi\left(\frac{1}{2^m+1}\right)\right) \leq$$
$$\leq 2^{-nk}A_{10}(1+A_{11})\sum_{m=1}^{n} 2^{mk}\varphi\left(\frac{1}{2^m+1}\right) = 2^{-nk}A_{12}\sum_{m=0}^{n} 2^{mk}\varphi\left(\frac{1}{2^m+1}\right) \leq$$

$$\leq 2^{-nk} A_{12} A_{13} \sum_{m=0}^{n} 2^{mk} \varphi\left(\frac{1}{2^m}\right) = 2^{-nk} A_{14} \sum_{m=0}^{n} 2^{mk} \varphi\left(\frac{1}{2^m}\right).$$

Since $2^{mk} \leq \int\limits_{m}^{m+1} 2^{tk} dt$ for $t \in (m, m+1)$ the following relation holds: $\frac{1}{2^{m+1}} < \frac{1}{2^t} < \frac{1}{2^m}$. For $t = -\log_2 \theta$ and according to the properties of φ

$$\varphi\left(\frac{1}{2^m}\right) = \varphi\left(\frac{1}{2^{m+1}}\right) \leq A_{15}\varphi\left(\frac{1}{2^{m+1}}\right) \leq A_{15}A_{16}\varphi\left(\frac{1}{2^t}\right) = A_{17}\varphi\left(\frac{1}{2^t}\right),$$

we have:

$$\omega_k\left(f, \frac{1}{2^n}\right) \leq 2^{-nk} A_{14} \sum_{m=0}^{n} 2^{mk} \varphi\left(\frac{1}{2^m}\right) \leq$$

$$\leq 2^{-nk} A_{14} A_{17} \sum_{m=0}^{n} \int_{m}^{m+1} 2^{mt} \varphi\left(\frac{1}{2t}\right) dt = 2^{-nk} A_{18} \int_{0}^{n+1} 2^{mt} \varphi\left(\frac{1}{2^t}\right) dt =$$

$$= 2^{-nk} \frac{A_{18}}{\ln 2} \int\limits_{\frac{1}{2^{n+1}}}^{1} \frac{\varphi(\theta)}{\theta^{k+1}} d\theta = 2^{-(n+1)k} A_{19} 2^k \int\limits_{\frac{1}{2^{n+1}}}^{1} \frac{\varphi(\theta)}{\theta^{k+1}} d\theta =$$

$$= 2^{-(n+1)k} A_{20} \int\limits_{\frac{1}{2^{n+1}}}^{1} \frac{\varphi(\theta)}{\theta^{k+1}} d\theta \leq A_{20} A_{21} \varphi\left(\frac{1}{2^{n+1}}\right) \leq A_{20} A_{21} A_{23} \varphi\left(\frac{1}{2^n}\right) =$$

$$= A_{24} \varphi\left(\frac{1}{2^n}\right).$$

Since for every $a \in (0,1)$ there exists a natural number n such that $\frac{1}{2^n} \leq a \leq \frac{2}{2^n}$, we have that

$$\omega_k(f, a)_p \leq \omega_k\left(f, \frac{1}{2^{n-1}}\right)_p \leq 2^k \omega_k\left(f, \frac{1}{2^n}\right)_p \leq 2^k A_{24} \varphi\left(\frac{1}{2^n}\right) \leq$$
$$\leq 2^k A_{24} A_{25} \varphi(\alpha) = A_{26} \varphi(\alpha)$$

respectively $f \in H(p, k, \varphi)$, $p \in [1, \infty]$, where $A_i, i = 1, 2, ..., 26$ are constants not dependent on θ. □

Theorem 2. *The function* $f(x) \backsim \sum_{m=-\infty}^{\infty} C_m e^{imx}$ *belongs to the class* $H(p, k, \varphi)$, $p \in [1, \infty]$ *if for its Fourier coefficients the following conditions are satisfied:*

i) $$\frac{\sqrt[4]{\sum\limits_{|m|=1}^{n}|m|^{4k}\sum\limits_{|m|=1}^{n}|C_m|^4}}{n^k}\le A_1\varphi\left(\frac{1}{n+1}\right)\text{ and}$$

$$\sqrt{\sum_{|m|=n+1}^{\infty}|C_m|^2}\le A_2\varphi\left(\frac{1}{n+1}\right),\text{ for } p\in[1,2].$$

ii) $$\sqrt[2p]{\sum_{|m|=1}^{n}|m|^{2p(k+1)-4}\sum_{|m|=1}^{n}|C_m|^{2p}}\le A_1\varphi\left(\frac{1}{n+1}\right)\text{ and}$$

$$\sqrt[p]{\sum_{|m|=n+1}^{\infty}|m|^{p-2}|C_m|^p}\le A_2\varphi\left(\frac{1}{n+1}\right),\text{ for } p\in[2,\infty).$$

iii) $$\sqrt{\sum_{|m|=1}^{n}|m|^{2k}\sum_{|m|=1}^{n}|C_m|^2}\le A_1\varphi\left(\frac{1}{n+1}\right)\text{ and}$$

$$\sum_{|m|=n+1}^{\infty}|C_m|\le A_2\varphi\left(\frac{1}{n+1}\right),\text{ for } p=\infty.$$

Proof. i) According to Prop. 2.2, using Cauchy-Bunjakowsky-Schwarz inequality and definition of the class $H(p,k,\varphi)$ for $p\in[1,2]$, we have:

$$\omega_k\left(f,\frac{1}{n}\right)\le A\left\{\frac{\sqrt{\sum\limits_{|m|=1}^{n}|m|^{2k}|C_m|^2}}{n^k}+\sqrt{\sum_{|m|=n+1}^{\infty}|C_m|^2}\right\}\le$$

$$\le A\left\{\frac{\sqrt[4]{\sum\limits_{|m|=1}^{n}|m|^{4k}\sum\limits_{|m|=1}^{n}|C_m|^4}}{n^k}+\sqrt{\sum_{|m|=n+1}^{\infty}|C_m|^2\}}\right\}\le$$

$$\le A\left\{A_1\varphi\left(\frac{1}{n+1}\right)+A_2\varphi\left(\frac{1}{n+1}\right)\right\}\le$$

$$\le A(A_1+A_2)\varphi(\frac{1}{n+1})=A_3\varphi\left(\frac{1}{n+1}\right)\Rightarrow f(x)\in H(p,k,\varphi),p\in[1,2]$$

where A, A_1, A_2, A_3 are constants not dependent on $f(x)$ and n. Conditions *ii)* and *iii)* can be proved analogously. □

Theorem 3. *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, be Fourier coefficients of a function $f(x)\in L_p$, $p\geq 1$. If the function $f(x)$ belongs to the class $H(p,k,\varphi)$, $p\in[1,\infty]$, then for its Fourier coefficients the following condition is satisfied:*

$$\frac{\sum\limits_{m=n+1}^{2n}(a_m+b_m)}{\sqrt[p]{n}}\leq A\varphi\left(\frac{1}{n+1}\right),$$

where A is a constant not dependent on $f(x)$ and n.

Proof. Since $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Fourier coefficients of a function $f(x)\in L_p$, $p\geq 1$, $f(x)$ belongs to the class $H(p,k,\varphi)$, $p\in[1,\infty]$ and according to relation (2.3) we have:

$$\frac{\left|\sum\limits_{m=n+1}^{2n}(a_m+b_m)\right|}{\sqrt[p]{n}}\leq A_1E_n(f)_p\leq A_1A_2\omega_k\left(f,\frac{1}{n}\right)_p\leq$$

$$\leq A_1A_2\omega_k\left(f,\frac{2}{n+1}\right)_p\leq A_1A_22^k\omega_k\left(f,\frac{1}{n+1}\right)_p=A_3\omega_k\left(f,\frac{1}{n+1}\right)_p\leq$$

$$\leq A_3A_4\varphi\left(\frac{1}{n+1}\right)=A_5\varphi\left(\frac{1}{n+1}\right)\Rightarrow\frac{\left|\sum\limits_{m=n+1}^{2n}(a_m+b_m)\right|}{\sqrt[p]{n}}\leq A_5\varphi\left(\frac{1}{n+1}\right),$$

where A_i, $i=1,2,\ldots,5$ are constants not dependent on $f(x)$ and n. □

Theorem 4. *Let the quasi-monotone sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be Fourier coefficients of a function $f(x)\in L_p$, $p\geq 1$. If the function $f(x)$ belongs to the class $H(p,k,\varphi)$, $p\in[1,\infty]$, then for its Fourier coefficients the following condition is satisfied:*

$$\sqrt[p]{n^{p-1}}(a_{2n}+b_{2n})\leq A\varphi\left(\frac{1}{n+1}\right),$$

where A is constant dependent on $\tau>0$ defined in the relation (1.1).

Proof. Since the Fourier coefficients of the function $f(x)\in L_p$, $p\geq 1$, $\{a_n\}$ and $\{b_n\}$ are quasi-monotone, then for $m<2n$ there exists constant A_1 such that:

$$(2n)^{-\tau}a_{2n}\leq A_1m^{-\tau}a_m\ \text{ and }\ (2n)^{\tau}b_{2n}\leq A_1m^{-\tau}b_m.$$

Adding these inequalities we have:

$$\begin{aligned}
&(2n)^{-\tau}(a_{2n}+b_{2n}) \leq A_1 m^{-\tau}(a_m+b_m)\\
&\Rightarrow m^{-\tau}(a_m+b_m) \geq \tfrac{1}{A_1}(2n)^{-\tau}(a_{2n}+b_{2n)}\\
&\Rightarrow a_m+b_m \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})m^{\tau},
\end{aligned}$$

considering that $0 < A_2 < 1$. From the system of inequalities

$$\begin{aligned}
&a_{n+1}+b_{n+1} \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})(n+1)^{\tau} \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})n^{\tau}\\
&a_{n+2}+b_{n+2} \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})(n+2)^{\tau} \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})n^{\tau}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{2n}+b_{2n} \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})(2n)^{\tau} \geq A_2(2n)^{-\tau}(a_{2n}+b_{2n})n^{\tau}
\end{aligned}$$

we get that

$$\begin{aligned}
&\sum_{m=n+1}^{2n}(a_m+b_m) \geq nA_2(2n)^{-\tau}(a_{2n}+b_{2n})n^{\tau} =\\
&A_2 2^{-\tau}n(a_{2n}+b_{2n}) = A_3 n(a_{2n}+b_{2n})
\end{aligned}$$

and respectively, $a_{2n}+b_{2n} \leq \dfrac{\sum\limits_{m=n+1}^{2n}(a_m+b_m)}{A_3 n}$. Now using the last inequality, since $f(x)$ belongs to the class $H(p,k,\varphi)$, $p \in [1,\infty]$ and according to the relation (2.3) we have:

$$a_{2n}+b_{2n} \leq \frac{\sum\limits_{m=n+1}^{2n}(a_m+b_m)}{A_3 n} \leq \frac{A_4\sqrt[p]{n}}{A_3 n}E_n(f)_p = \frac{A_5}{\sqrt[p]{n^{p-1}}}E_n(f)_p$$

$$\begin{aligned}
&\Rightarrow \sqrt[p]{n^{p-1}}(a_{2n}+b_{2n}) \leq A_5 E_n(f)_p \leq A_6\omega_k\left(f,\tfrac{1}{n}\right)_p \leq\\
&\quad \leq A_6A_7\omega_k\left(f,\frac{2}{n+1}\right)_p \quad \leq A_7 2^k\omega_k\left(f,\tfrac{1}{n+1}\right)_p \leq A_8\omega_k\left(f,\tfrac{1}{n+1}\right)_p \leq\\
&\quad \leq A_8A_9\varphi\left(\frac{1}{n+1}\right) = A_{10}\varphi\left(\frac{1}{n+1}\right)\\
&\Rightarrow \sqrt[p]{n^{p-1}}(a_{2n}+b_{2n}) \leq A_{10}\varphi\left(\frac{1}{n+1}\right),
\end{aligned}$$

where $A_i, i = 1,2,\ldots,10$, are constants dependent on τ, which is defined in the relation (1.1). □

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