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ANALYTIC REPRESENTATION OF A PRIMITIVE DISTRIBUTION

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Abstract

In this work we give a theorem for analytic representation of a primitive distribution for a given distribution. We give two examples.

The symbols used here are commonly adopted in the theory of distributions: $D(R)$ is the space of the test functions of R, and D' is the space of the distribution.

An important operation with the distributions is their analytic representation. Namely, if $T \in D'$, then the following result is correct: there are a pair of functions $f_+(z)$ and $f_-(z)$ analytic in the upper half plane Π^+, i.e. in the lower half plane Π^- from C respectively, such that the regular distributions $f_+(x+i\varepsilon) - f_-(x-i\varepsilon)$ converge towards T when $\varepsilon \to 0^+$. Thus:

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} [f_+(x+i\varepsilon) - f_-(x-i\varepsilon)]\varphi(x)dx = T(\varphi) \quad \varphi \in D$$

(1, p. 76).

The function f_+ is called upper function, f_- is the lower function, and both the functions together are an analytic representation for the distribution T. Otherwise, besides $T(\varphi)$, we will also write $\langle T, \varphi \rangle = T(\varphi)$. The analytic representation for a same distribution is unique for the up to entire function. Each entire function is an analytic representation for the zero distribution O.

If the complement of the support from the distribution is not an empty set in R, then f_+ and f_- of the complement are an analytic continuation f, one on another, and the continued function in this way is also an analytic representation for the distribution T. Further on, if $f(z)$ is an analytic representation for T, $f'(z)$ is an analytic representation for T' and in general, $f^{(m)}$ is an analytic representation for the m-th derivative $T^{(m)}$ of the distribution T.

Determining the analytic representation for a given distibution T, generally is not an easy task. However, if $T \in O'_\alpha$, where $\alpha \leq -1$, O_α are spaces from functions particularly adapted for the determination of the analytic representation ([1], p. 81), then the analytic representation for T is the function

$$\hat{T}(z) = \frac{1}{2\pi i}\langle T, \frac{1}{t-z}\rangle, \quad \operatorname{Im} z \neq 0, \quad z = x + iy$$

$\hat{T}(z)$ is also called Couchy representation, and it is always analytic to C except to the supporter $\operatorname{supp} T$ of the distribution T.

For the derivative T' the analytic representation is

$$\hat{T}(z) = \frac{1}{2\pi i}\langle T, \frac{+1}{(t-z)^2}\rangle$$

and generally for $T^{(m)}$,

$$\hat{T}^{(m)}(z) = \frac{m!}{2\pi i}\langle T, \frac{1}{(t-z)^{m+1}}\rangle .$$

Here, we present a theorem for the analytic representation of the primitive distribution S (or distribution integral) for a given distribution T. The distribution S is an integral for the distribution T, if it is true that $S' = T$. Or, more generally, if the m-th derivative $S^{(m)} = T$, then S is called m-multiple integral for the distribution T. The m-multiple integral exists always and it is solely determined up to polynomial from degree $m - 1$. ([2], p. 94).

Theorem. If the pair $f_+(z)$, $f_-(z)$ is an analytic representation for the distribution T, then the individual primitive functions $F_+(z)$, $F_-(z)$ are an analytic representation for the primitive distribution S for T.

Proof. The functions $F_+(z)$, $F_-(z)$ exist because the half planes Π^+, Π^- are simply connect domain in C. Let the support $\operatorname{supp} T \neq R$. Since $\operatorname{supp} T$ is a closed set in R, the complement $R \setminus \operatorname{supp} T = \Omega$ is an open set.

Let the function $\rho(t) \in D(\Omega)$ be such that $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$. In that case each function $\varphi \in D$ can uniquely be presented in the form

$$\varphi(t) = a\rho(t) + \varphi^*(t), \quad a = \int_{-\infty}^{\infty} \varphi(t)\,dt \quad \text{and} \quad \int_{-\infty}^{\infty} \varphi^*(\tau)d\tau \in D\,.$$

With the relation

$$\langle S, \varphi\rangle = -\langle T, \int_{\infty}^{t} \varphi^*(\tau)d\tau\rangle \tag{1}$$

the primitive distribution S is determined ([2], p.96).

$$\int_{-\infty}^{\infty} [F_+(x+i\varepsilon) - F_-(x-i\varepsilon)]\varphi(x)dx = \int_{-\infty}^{\infty} [F^+(x+i\varepsilon)$$

$$- F^-(x-i\varepsilon)]a p(x)dx + \int_{-\infty}^{\infty} [F_+(x+i\varepsilon) - F_-(x-i\varepsilon)]\varphi^*(x)dx$$

$$\lim_{\varepsilon\to 0} a \int_{-\infty}^{\infty} [F_+(x+i\varepsilon) - F_-(x+i\varepsilon)]\rho(x)dx = 0$$

because to Ω one function is an analytic continuation of the other, the integral $\int_{-\infty}^{\infty} [F_+(x+i\varepsilon) - F_-(x-i\varepsilon)]\varphi^*(x)dx$, with partial integration is

$$[F_+(x+i\varepsilon) - F_-(x-i\varepsilon)] \int_{-\infty}^{x} \varphi^*(t)\,dt\Big|_{-\infty}^{\infty} -$$

$$- \int_{-\infty}^{\infty} [f^+(x+i\varepsilon) - f^-(x-i\varepsilon)] \int_{-\infty}^{x} \varphi^*(t)\,dt$$

$\int_{-\infty}^{\infty} \varphi^*(t)\,dt = 0$ and those is why we have

$$\int_{-\infty}^{\infty} [F_+(x+i\varepsilon) - F_-(x-i\varepsilon)]\varphi(x)dx = - \int_{-\infty}^{\infty} [f^+(x+i\varepsilon) - f^-(x-i\varepsilon)] \int_{-\infty}^{x} \varphi^*(t)\,dt$$

Taking the limes when $\varepsilon \to 0^+$ we have

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}[F_+(x+i\varepsilon)-F_-(x-i\varepsilon)]\varphi(x)dx =$$

$$= -\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}[f_+(x+i\varepsilon)-f_-(x-i\varepsilon)]\int_{-\infty}^{x}\varphi^*(t)\,dt$$

$$= -\langle T, \int_{-\infty}^{x}\varphi^*(t)\,dt\rangle = S(\varphi).$$

By this we conclude this proof. Because the primitive distribution S for a given distribution T is unique up to a constant distribution $[c]$, i.e. every distribution $S+[c]$ is primitive for T, that is why the analytic representation of S is unique to the analytic representation of the constant $[c]$, whose representation is c for $\operatorname{Im} z > 0$ and 0 for $\operatorname{Im} z < 0$, or $\frac{c}{2}$ for $\operatorname{Im} z > 0$ and $-\frac{c}{2}$ for $\operatorname{Im} z < 0$.

From here it follows that the determining of an analytic representation of a primitive distribution for a specified distribution will be obtained by an appropriate choice of the constant c.

By analogy, if the distribution S is m-multiple integral for the distribution T: $S^{(m)} = T$, then every distribution in the form of $S + [P]$, where P is polynomial of a degree $m-1$, is m-multiple integral for T. From here it follows that the analytic representation of S shall be unique up to a representation of a polynomial

$$P(t) = a_{m-1}t^{m-1} + \cdots + a_0,$$

whose representation is $a_{m-1}z^{m-1}+\cdots+a_0$ for $\operatorname{Im} z > 0$ and 0 for $\operatorname{Im} z < 0$.

Examples:

The integral for the δ distribution is the distribution $S = [H] + [c]$, $H(t)$ is the Heaviside function. Because the function

$$f(z) = -\frac{1}{2\pi i}\frac{1}{z}, \qquad \operatorname{Im} z > 0$$

is analytic representation for the distribution δ, that is why, for the distribution S, the analytic representation is the pair $F_+(z)$, $F_-(z)$ where

$$F_+(z) = \frac{1}{2\pi i}\log z + \frac{c}{2} \quad \text{for} \quad \operatorname{Im} z > 0$$

$$F_-(z) = -\frac{1}{2\pi i}\log z - \frac{c}{2} \quad \text{for} \quad \operatorname{Im} z < 0.$$

From the condition for analytic representation for the distribution $[H]$ we get that $[c] = [1]$, and in consequence, the representation shall be:

$$[\hat{H}](z) = \begin{cases} -\dfrac{1}{2\pi i}\log z + \dfrac{1}{2} & \operatorname{Im} z > 0 \\ -\dfrac{1}{2\pi i}\log z - \dfrac{1}{2} & \operatorname{Im} z > 0 \end{cases}$$

Because of the relations

$$-\frac{1}{2\pi i z} + \frac{1}{2} = -\frac{1}{2\pi i}\log(-z) \quad \text{for} \quad \operatorname{Im} z > 0$$

$$-\frac{1}{2\pi i z} - \frac{1}{2} = -\frac{1}{2\pi i}\log(-z) \quad \text{for} \quad \operatorname{Im} z < 0$$

we can put

$$[\hat{H}](z) = -\frac{1}{2\pi i}\log(-z) \quad \operatorname{Im} z \neq 0.$$

By analogy, for the distribution $H_-(t) = [H(-t)]$ we get

$$[\hat{H}_-](z) = \frac{1}{2\pi i}\log z, \quad \operatorname{Im} z \neq 0.$$

2. The primitive distribution for the distribution $T = [H(t)] + [c]$ is the distribution $S = [tH(t)] + [ct + d]$.

From the already described method we get that analytic representation is the function

$$-\frac{1}{2\pi i} z\log(-z) + cz + d \quad \operatorname{Im} z > 0$$

$$-\frac{1}{2\pi i} z\log(-z) + 0, \quad \operatorname{Im} z < 0.$$

For the distribution $[tH(t)]$ the representation shall be the function

$$-\frac{1}{2\pi i} z\log(-z) \quad \operatorname{Im} z \neq 0.$$

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АНАЛИТИЧКА РЕПРЕЗЕНТАЦИЈА НА ПРИМИТИВНАТА ДИСТРИБУЦИЈА

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Резиме

Во оваа работа е дадена теорема за аналитична репрезентација на примитивна дистрибуција за дадена дистрибуција. Дадени се два примери.

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