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ON SOME CLASSICAL INEQUALITIES IN UNITARY SPACES

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Abstract. In this paper we shall give some remarks about some classical inequalities in unitary spaces. In the first section we shall consider some inequalities which are in connectior to the Buniakowsky-Cauchy-Schwarz inequality, in the second to the Bessel inequality, and in the third to the Clarkson inequality.

1. On some inequalities related to the Buniakowsky-Cauchy-Schwarz inequality

S. Kurepa [1] proved the following two results:

Theorem A. Let X be a real Hilbert space and X_C the complexfication of X. Then for any couple of vectors $a \in X$ and $z \in X_C$ the following inequality holds:

$$|(z,a)|^2 \le |a|^2\left(\tfrac{1}{2}|z|^2 + \tfrac{1}{2}|(z,\bar{z})|\right) \le |a|^2|z|^2 \tag{1}$$

where $\bar{z}$ denotes the conjugate vector of z.

Theorem B. Let X be a real Hilbert space and e a unit vector in X, suppose that $a,b \in X$ are given vectors such that

$$(u^2-|a_0|^2) + (v^2-|b_0|^2) \ge 0 \tag{2}$$

where $u=(a,e)$, $v=(b,e)$, $a_0=a-ue$, $b_0=b-ve$.

Then the following inequality holds:

$$(u^2-|a_0|^2)(v^2-|b_0|^2) \le (uv-(a_0,b_0))^2. \tag{3}$$

If a and b are independent vectors and in (2) the strict inequality holds, then also in (3) strict inequality holds.

Here we shall give some related results.

Our first result is an extension of a result from [2], and it is related to (1).

Theorem 1. Let X be a real unitary space and $f,g,h\in X$. Then

$$-\frac{1}{2}|f|^2(|g||h|-(g,h)) \le (f,g)(f,h) \le \frac{1}{2}|f|^2(|g||h|+(g,h)). \tag{4}$$

Proof. If g and $h\neq 0$ we can suppose that $|g|=|h|$. Further let $g=p+q$, $h=p-q$, i.e. $p = \frac{g+h}{2}$, $q = \frac{g-h}{2}$. Then we have $(p,q)=0$. We also have

$$|g||h| = |p+q||p-q| = |p|^2 + |q|^2,$$

and

$$(g,h) = (p+q,p-q) = |p|^2 - |q|^2.$$

Suppose that $r=f-up-vq$, where u and v are real numbers such that

$$(r,p) = 0 \quad \text{and} \quad (r,q) = 0.$$

Then

$$|f|^2 = (f,f) = |r|^2 + u^2|p|^2 + v^2|q|^2,$$

so we have

$$(f,g)(f,h) = u^2|p|^4 - v^2|q|^4.$$

Therefore

$$(f,g)(f,h) \le u^2|p|^2|p|^2 \le \frac{1}{2}|f|^2(|g||h|+(g,h))$$

and

$$(f,g)(f,h) \ge -v^2|q|^2|q|^2 \ge -\frac{1}{2}|f|^2(|g||h|-(g,h)).$$

In connection to Theorem B is:

Theorem 2. If X is a real Hilbert space, $x_1,\dots,x_m$ and $y_1,\dots,y_m$ are vectors from X and u,v are real numbers such that we have

$$u_2 - G(x_1,\dots,x_m) > 0 \text{ or } v^2 - G(y_1,\dots,y_m) > 0, \tag{5}$$

where G is the Gram determinant, then

$$(u^2-G(x_1,\dots,x_m))(v^2-G(y_1,\dots,y_m)) \le \tag{6}$$

$$\le \left\{uv-\det\begin{bmatrix}(x_1,y_1) & \dots & (x_1,y_m)\\ \vdots & & \\ (x_m,y_1) & \dots & (x_m y_m\end{bmatrix}\right\}^2.$$

Proof. Let $e_1,\dots,e_m$ be any orthonormal set in X. Using Lemma 3 from [1] and the well-known Aczél inequality (see for example [3, p. 57]) we get

$$(u^2-G(x_1,\dots,x_m))(v^2-G(y_1,\dots,y_m)) =$$

$$= \left(u^2- \sum_{j_1<\dots<j_m} \left|\det\begin{bmatrix}(x_1,e_{j_1})\dots(x_1,e_{j_m})\\ \cdot \quad \dots \quad \cdot \\ (x_m,e_{j_1})\dots(x_m,e_{j_m})\end{bmatrix}\right|^2\right) \times$$

$$\times \left(v^2- \sum_{j_1<\dots<j_m} \left|\det\begin{bmatrix}(y_1,e_{j_1})\dots(y_1,e_{j_m})\\ \cdot \quad \dots \quad \cdot \\ (y_m,e_{j_1})\dots(y_m,e_{j_m})\end{bmatrix}\right|^2\right) \le$$

$$\le \left(uv- \sum_{j_1<\dots<j_m} \det\begin{bmatrix}(x_1,e_{j_1})\dots(x_1,e_{j_m})\\ \cdot \quad \dots \quad \cdot \\ (x_m,e_{j_1})\dots(x_m,e_{j_m})\end{bmatrix}\det\begin{bmatrix}(e_{j_1},y_1)\dots(e_{j_m},y_m)\\ \cdot \quad \dots \quad \cdot \\ (e_{j_m},y_1)\dots(e_{j_m},y_m)\end{bmatrix}\right)^2 =$$

$$= \left\{uv - \det\begin{bmatrix}(x_1,y_1)\dots(x_1,y_m)\\ \cdot \quad \dots \quad \cdot \\ (x_m,y_1)\dots(x_m,y_m)\end{bmatrix}\right\}^2.$$

Moreover, if we use Aczél's inequality for n=2 with $a_1 \to u$, $b_1 \to v$, $a_2^2 \to G(x_1,\dots,x_m)$, $b_2^2 \to G(y_1,\dots,y_m)$, we get:

<u>Theorem 3</u>. If X is a real Hilbert space, $x_1,\dots,x_m$ and $y_1,\dots,y_m$ are vectors from X and u,v are real numbers such that

$$u^2 - G(x_1,\dots,x_m) > 0 \text{ and } v^2 - G(y_1,\dots,y_m) > 0, \qquad (7)$$

then

$$(u^2-G(x_1,\dots,x_m))(v^2-G(y_1,\dots,y_m)) \le$$
$$\le (uv-G^{1/2}(x_1,\dots,x_m)G^{1/2}(y_1,\dots,y_m))^2 \le$$
$$\le \left(uv-\det\begin{bmatrix}(x_1,y_1)\dots(x_1,y_m)\\ \vdots \quad \dots \quad \cdot \\ (x_m,y_1)\dots(x_m,y_m)\end{bmatrix}\right)^2. \qquad (8)$$

Of course, in the last inequality we have used Corollary 5 from [1].

The following generalization of a result from [4, p. 383] is given in [5]:

<u>Theorem C</u>. Let a and b be lineary independent vectors in an unitary vector space V and let x be a vector in V such that

$$(x,a) = u \quad \text{and} \quad (x,b) = v. \tag{9}$$

Then

$$G(a,b)|x|^2 \geq |\bar{u}b - \bar{v}a|^2 \tag{10}$$

with equality if and only if

$$x = \frac{(a,\bar{v}a-\bar{u}b)b-(b,\bar{v}a-\bar{u}b)a}{G(a,b)}. \tag{11}$$

Here we shall note that the following generalization of results from [4, p. 383], [5] and [6] can be proved similarly:

Theorem 4. Let $a,b,c_1,\ldots,c_n$ be vectors in an unitary vector space V such that $(a,c_j)(b,c_i) \neq (a,c_i)(b,c_j)$ for $i\neq j$. If p_{ij} $(i,j=1,\ldots,n,\ i\neq j)$ are real numbers such that $p_{ij} = p_{ji}$, $P = \sum_{1\leq i<j\leq n} p_{ij} \neq 0$ then

$$P^2\frac{|va-ub|^2}{G(a,b)} \leq \sum_{i=1}^{n} \Big|\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{p_{ij}(va-ub,c_j)}{(a,c_j)(b,c_i)-(a,c_i)(b,c_j)}\Big|^2. \tag{12}$$

Moreover, further generalizations of these results can be given in the form as in [7]. For given matrix $A=[a_{ij}]$ we use notation $A^{(p)}$ for the matrix $[a_{ij}{}^p]$. We have

$$G^p(a,b) = (|a|^2|b|^2-|(a,b)|^2)^p \leq |a|^{2p}|b|^{2p}-|(a,b)|^{2p} = G^{(p)}(a,b),$$

as a consequence of the following simple inequality

$$(a-b)^p < a^p-b^p \quad (a > b > 0,\ p > 1).$$

(In fact, this inequality is a simple consequence of Petrović's inequality [3, p. 23] for convex function $f(x) = x^p$, $p > 1$.)

Using this result and Theorems C and 4 we get:

Theorem 5. Let the conditions of Theorem C be fulfilled and let $p \geq 1$. Then

$$G^{(p)}(a,b)|x|^{2p} \geq |\bar{u}b-\bar{v}a|^{2p}. \tag{13}$$

Theorem 6. Let the conditions of Theorem 4 be fulfilled and let $p \geq 1$. Then

$$P^{2p}\frac{|va-ub|^{2p}}{G^{(p)}(a,b)} \leq \Big(\sum_{i=1}^{n} \Big|\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{p_{ij}(va-ub,c_j)}{(a,c_j)(b,c_i)-(a,c_i)(b,c_j)}\Big|^2\Big)^p.$$

Note that in [7] a discrete analogue of (13) for u=0, v=1 and $p\in N$ was considered. The above proof is simpler than that from [7].

2. On some generalizations of the Bessel inequality

E. Bombieri [8] proved the following generalization of the well-known Bessel inequality:

Theorem D. If $x, y_1, \ldots, y_n$ are elements of an unitary space over the field of complex numbers, then

$$\sum_{r=1}^{n} |(x,y_r)|^2 \leq |x|^2 \max_{1\leq r\leq n} \sum_{s=1}^{n} |(y_r,y_s)|. \tag{15}$$

We see that if the y_r are orthonormal then the above reduces to the Bessel inequality.

In fact Bombieri's result is equivalent to ([9]):

Theorem E. If $x, y_1, \ldots, y_n$ are as above, then

$$\left|\sum_{r=1}^{n} C_r(x,y_r)\right|^2 \leq \left(\sum_{r=1}^{n} |C_r|^2\right)|x|^2 \max_{1\leq r\leq n} \sum_{s=1}^{n} |(y_r,y_s)|, \tag{16}$$

where C_r are arbitrary complex numbers.

A. Selberg [10] proved the following generalization of the Bessel inequality:

Theorem F. If $x, y_1, \ldots, y_n$ are as above then

$$\sum_{r=1}^{n} |(x,y_r)|^2 \left(\sum_{s=1}^{n} |(y_r,y_s)|\right)^{-1} \leq |x|^2. \tag{17}$$

H. Heilbronn [11] proved:

Theorem G. If $x, y_1, \ldots, y_n$ are as above, then

$$\sum_{r=1}^{n} |(x,y_r)| \leq |x| \left(\sum_{r,s} |(y_r,y_s)|\right)^{1/2}. \tag{18}$$

Here, we shall note that the following interpolation of (16) is valid:

Theorem 7. Let the conditions of Theorem E be fulfilled. Then

$$\left|\sum_{r=1}^{n} c_r(x,y_r)\right|^2 \le |x|^2 \sum_{r=1}^{n} |c_r|^2 \sum_{s=1}^{n} |(y_r,y_s)| \le$$
$$\le |x|^2 \left(\sum_{r=1}^{n} |c_r|^2\right) \max_{1\le r\le n} \sum_{s=1}^{n} |(y_r,y_s)|. \tag{19}$$

<u>Proof</u>. Since

$$\sum_{r=1}^{n} c_r(x,y_r) = \left(x, \sum_{r=1}^{n} \bar{c}_r y_r\right),$$

we get

$$\left|\sum_{r=1}^{n} c_r(x,y_r)\right|^2 \le |x|^2 \left|\sum_{r=1}^{n} \bar{c}_r y_r\right|^2. \tag{20}$$

Further, we have

$$\left|\sum_{r=1}^{n} \bar{c}_r y_r\right|^2 = \sum_{r,s=1}^{n} \bar{c}_r c_s (y_r,y_s) \le \sum_{r,s=1}^{n} |c_r||c_s||(y_r,y_s)|. \tag{21}$$

Since $|c_r||c_s| \le \frac{1}{2}(|c_r|^2+|c_s|^2)$, we get from (21)

$$\left|\sum_{r=1}^{n} \bar{c}_r y_r\right|^2 \le \sum_{r=1}^{n} |c_r|^2 \sum_{s=1}^{n} |(y_r,y_s)| \le \left(\sum_{r=1}^{n} |c_r|^2\right) \max_{1\le r\le n} \sum_{s=1}^{n} |(y_r,y_s)|. \tag{22}$$

Now, (20) and (22) give (19).

<u>Remark</u>. In fact, we proved

$$\left|\sum_{r=1}^{n} c_r(x,y_r)\right|^2 \le |x|^2 \left|\sum_{r=1}^{n} \bar{c}_r y_r\right|^2 \le |x|^2 \sum_{r=1}^{n} |c_r|^2 \sum_{s=1}^{n} |(y_r,y_s)| \le$$
$$\le |x|^2 \left(\sum_{r=1}^{n} |c_r|^2\right) \max_{1\le r\le n} \sum_{s=1}^{n} |(y_r,y_s)|.$$

Now, we shall show that Theorems F and G are simple consequences of the first inequality in (19). Indeed, for

$$c_r = \overline{(x,y_r)}\left(\sum_{s=1}^{n} |(y_r,y_s)|\right)^{-1}$$

we get

$$\left(\sum_{r=1}^{n} |(x,y_r)|^2 \left(\sum_{s=1}^{n} |(y_r,y_s)|\right)^{-1}\right)^2 \le$$
$$\le |x|^2 \sum_{r=1}^{n} |(x,y_r)|^2 \left(\sum_{s=1}^{n} (y_r,y_s)|\right)^{-1},$$

what is equivalent to (17).

Now, for

$$c_r = \exp(-i \arg(x,y_r))$$

the first inequality in (19) becomes

$$\left(\sum_{r=1}^{n} |(x,y_r)|\right)^2 \leq |x|^2 \sum_{r=1}^{n} \sum_{s=1}^{n} |(y_r,y_s)|,$$

i.e. (18).

3. Inequalities of Clarkson type

Let $(X,(.,.))$ be a prehilbertian space on K $(K=C,R)$. The following results are proved by S.S. Dragomir and I.Sandor [12]:

For every $x,y \in X$ we have

$$|x+y|^p + |x-y|^p \geq (|x|+|y|)^p + ||x|-|y||^p \qquad (23)$$

if $1 < p < 2$, and

$$|x+y|^p + |x-y|^p \geq 2(|x|^p + |y|^p) \qquad (24)$$

if $p \geq 2$.

In fact, using the idea of their proof we can prove the following result:

Theorem 8. Let $(X,(.,.))$ be a real or complex prehilbertian space and let $x,y \in X$. If $0 < p \leq 2$ we have

$$(|x|+|y|)^p + ||x|-|y||^p \leq |x+y|^p + |x-y|^p \leq 2(|x|^2+|y|^2)^{p/2}. \qquad (25)$$

If either $p \geq 2$ or $p < 0$, the nequalities in (25) are reversed.

Now, using the classical Clarkson lemma for real numbers (see for example [13]) we can obtain the Clarkson inequalities for unitary spaces:

Theorem 9. Let X be an unitary space and let $x,y \in X$. If $1 < p \leq 2$, then $(p^{-1}+q^{-1}=1)$:

$$|x+y|^q + |x-y|^q \leq 2(|x|^p + |y|^p)^{q-1}, \qquad (26)$$

and

$$|x+y|^p + |x-y|^p \geq 2(|x|^q + |y|^q)^{p-1}. \qquad (27)$$

If $p \geq 2$ the inequalities are reversed.

The second inequality in (25) can be generalized for n vectors.

Of course in a prehilbertian space X we have a parallelogram identity

$$|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2). \tag{28}$$

By a method of mathematical induction we can prove the following generalization of this identity:

$$\Sigma|x_1 \pm x_2 \pm \ldots \pm x_n|^2 = 2^{n-1} \sum_{i=1}^{n} |x_i|^2, \tag{29}$$

where $x_i \in X$, $i=1,\ldots,n$, and the summation on the left-hand side is taken over all (2^{n-1}) possible choices of the ± signs. Of course (29) is equivalent to the following

$$\Sigma|\pm x_1 \pm x_2 \pm \ldots \pm x_n|^2 = 2^n \sum_{i=1}^{n} |x_i|^2, \tag{29'}$$

where we now have 2^n possible choices of the ± signs in summation.

Set

$$S_\lambda \equiv (\Sigma|\pm x_1 \pm x_2 \pm \ldots x_n|^\lambda)^{1/\lambda} \text{ and } Q_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}.$$

The following result is a generalization of the second inequality in (25) and of the results from [14], [15].

<u>Theorem 10</u>. Let X be an unitary space and let $x_1,\ldots,x_n \in X$. For $\lambda > 2$, we have

$$S_\lambda \geq 2^{n/\lambda} Q_2 \tag{30}$$

while for $\lambda < 2$ $(\neq 0)$ we have the reverse inequality. For $\lambda = 2$ we have the identity (29').

Also, for $\lambda \geq 2$, we have

$$S_\lambda^\lambda \geq 2^n \sum_{i=1}^{n} |x_i|^\lambda. \tag{31}$$

Further generalization of this result and of results from [16] can be given similarly to the proof from [16].

Theorem 11. Let X, $x_1, \ldots, x_n$ be as in Theorem 10. We have:

(i) If $p, \lambda \geq 2$, then

$$2^{n/\lambda} Q_p \leq S_\lambda \leq n^{1/2-1/p} 2^{(n-1)/2+1/\lambda} Q_p. \tag{32}$$

If $0 < p, \lambda \leq 2$, then the reverse inequalities are valid in (32).

(ii) If $0 < \lambda \leq 2$, $p \geq 2$, then

$$2^{(n-1)/2+1/\lambda} Q_p \leq S_\lambda \leq 2^{n/\lambda} n^{1/2-1/p} Q_p . \tag{33}$$

For $0 < p \leq 2$ and $\lambda \geq 2$ we have the reverse inequalities in (33).

(iii) For $\lambda > 0$, $p < 0$ we have

$$S_\lambda \geq 2^{(n-1)/2+1/\lambda} n^{1/2-1/p} Q_p. \tag{34}$$

(iv) For $\lambda > 2$, $p < 0$, we have

$$S_\lambda \geq 2^{n/\lambda} n^{1/2-1/p} Q_p. \tag{35}$$

For $\lambda < 0$ and $p > 2$ we have the reverse inequalities in (35).

(v) For $\lambda < 0$ and $0 < p < 2$ we have

$$S_\lambda \leq 2^{n/\lambda} Q_p. \tag{36}$$

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ЗА НЕКОИ КЛАСИЧНИ НЕРАВЕНСТВА ВО УНИТАРНИ ПРОСТОРИ

Јосип Е. Печариќ

Р е з и м е

Во трудот се дадени неколку забелешки за некои класични неравенства во унитарни простори, кои се поврзани со неравенството на Коши-Буњаковски-Шварц, Беселовото неравенство и неравенството на Кларксон.

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