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CERTAIN INEQUALITES FOR FINITE SUMS WITH THE FIBONACII NUMBERS

Mao Qi Ji * and Zdravko F. Starc **

Abstract

In this note certain inequalities for finite sums with the Fibonacci numbers are given.

0. The Fibonacci numbers F_n are defined by

$$F_1 = F_2 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad (n = 3, 4, \dots).$$

These numbers have many interesting properties, from which we cite only two:

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1\,, \tag{1}$$

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}\,. \tag{2}$$

In [3], using the Caushy inequality we have obtained the following inequality

$$a_1F_1 + a_2F_2 + \cdots + a_nF_n < 2^n\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

and some consequences.

In the first part of this note, we apply the Caushy inequality to create new inequalities for finite sums with the Fibonacci numbers. Some other inequalities we obtain in the second part of this note.

1. Now, we will proove the following results

Theorem 1. *The following inequalities*

$$\frac{a_1}{F_1^2}+\frac{a_2}{F_2^2}+\cdots+\frac{a_n}{F_n^2}>\left(\frac{3}{5}\right)^{2n}(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\,)^2\,, \tag{3}$$

$$\frac{a_1}{F_1}+\frac{a_2}{F_2}+\cdots+\frac{a_n}{F_n}>\left(\frac{3}{5}\right)^{n+1}(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\,)^2\,, \tag{4}$$

$$a_1\ln F_1+a_2\ln F_2+\cdots+a_n\ln F_n<\frac{1}{5}\sqrt{(n-1)n(2n-1)}\sqrt{a_1^2+a_2^2+\cdots+a_n^2}, \tag{5}$$

$$a_1\,e^{-F_1}+a_2\,e^{-F_2}+\cdots+a_n\,e^{-F_n}<2\sqrt{1-\left(\frac{3}{4}\right)^n}\sqrt{a_1^2+a_2^2+\cdots+a_n^2} \tag{6}$$

are hold where a_i $(i=1,2,\ldots,n)$ *are positive numbers and* $n=3,4,\ldots$.

Proof. Because of [4]

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}<F_n<\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\qquad(n=3,4,\ldots)$$

we obtain

$$\left(\frac{4}{3}\right)^{n-2}<F_n<\left(\frac{5}{3}\right)^{n-1} \tag{7}$$

or

$$F_n\,F_{n+1}<\left(\frac{5}{3}\right)^{2n},\quad\sqrt{F_n\,F_{n+1}}<\left(\frac{5}{3}\right)^{n}, \tag{8}$$

Putting $x_i=\dfrac{\sqrt{a_i}}{F_i}$, $y_i=F_i$ $\quad(i=1,2,\ldots,n)$ in the Caushy inequality

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\le(x_1^2+x_2^2+\cdots+x_n^2)(y_1^2+y_2^2+\cdots+y_n^2) \tag{9}$$

and by means of the equality (2) and the inequality (8), we obtain the inequality (3).

By the same arguments, putting $x_i=\sqrt{\dfrac{a_i}{F_i}}$, $y_i=\sqrt{F_i}$ $(i=1,2,\ldots,n)$ in the inequality (9) and by means of the equality (1) and the inequality (8) we obtain the inequality (4).

Putting $x_i = a_i$, $y_i = \ln F_i$ $(i = 1, 2, \ldots, n)$ in the inequality (9) and noticing that

$$\ln^2 F_1 + \ln^2 F_2 + \cdots + \ln^2 F_n < (1^2 + 2^2 + \cdots + (n-1)^2)\left[\ln\left(\frac{1+\sqrt{5}}{2}\right)\right]^2$$

and

$$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6}, \quad \left[\ln\left(\frac{1+\sqrt{5}}{2}\right)\right]^2 < 0,24$$

we obtain the inequality (5).

For $x_i = a_i$, $y_i = e^{-F_i}$ $(i = 1, 2, \ldots, n)$ by means of (7) and $e^{-x} < x^{-1}$ the inequality (9) becomes (6).

We will mention here three simple consequences of the above theorem.

Corollary 1. *In the case when* $a_i = 1$ $(i = 1, 2, \ldots, n)$ *the inequalities* (3)-(6) *become*

$$\frac{1}{F_1^2} + \frac{1}{F_2^2} + \cdots + \frac{1}{F_n^2} > \left(\frac{3}{5}\right)^{2n} n^2,$$

$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_n} > \left(\frac{3}{5}\right)^{n+1} n^2,$$

$$F_1 F_2 \ldots F_n < \exp \frac{1}{5} n\sqrt{(n-1)(2n-1)},$$

$$e^{-F_1} + e^{-F_2} + \cdots + e^{-F_n} < 2\sqrt{n}\sqrt{1 - \left(\frac{3}{4}\right)^n}$$

respectively.

Corollary 2. *In the case when* $a_i = i^2$ $(i = 1, 2, \ldots, n)$ *the inequalities* (3)-(6) *become*

$$\frac{1^2}{F_1^2} + \frac{2^2}{F_2^2} + \cdots + \frac{n^2}{F_n^2} > \left(\frac{3}{5}\right)^{2n} \frac{n^2(n+1)^2}{4},$$

$$\frac{1^2}{F_1} + \frac{2^2}{F_2} + \cdots + \frac{n^2}{F_n} > \left(\frac{3}{5}\right)^{n+1} \frac{n^2(n+1)^2}{4},$$

$$F_1 F_2^2 \ldots F_n^n < \exp \frac{n^3}{5},$$

$$e^{-F_1} + 2e^{-F_2} + \cdots + ne^{-F_n} < \frac{\sqrt{6}}{3}\sqrt{n(n+1)(2n+1)}\sqrt{1 - \left(\frac{3}{4}\right)^n}$$

respectively.

Corollary 3. *In the case when* $a_i = F_i$ $(i = 1, 2, \ldots, n)$ *the inequalities* (5)-(6) *become*

$$F_1^{F_1} F_2^{F_2} \ldots F_n^{F_n} < \exp \frac{\sqrt{2}}{5} \sqrt{n^3} \left(\frac{5}{3}\right)^n,$$

$$F_1 e^{-F_1} + F_2 e^{-F_2} + \cdots + F_n e^{-F_n} < 2\left(\frac{5}{3}\right)^n \sqrt{1 - \left(\frac{3}{4}\right)^n}$$

respectively.

2. Next we will prove the following results

Theorem 2. *The following inequality*

$$\sqrt{(F_1 + 2F_{2n})(2F_1 + F_{2n})} + \sqrt{(F_2 + 2F_{2n-1})(2F_n + F_{2n-1})} + \cdots +$$

$$+ \sqrt{(F_n + 2F_{n+1})(2F_n + F_{n+1})} \leq \frac{3}{2}(F_{2n+2} - 1)$$

holds where $n = 1, 2, \ldots$.

Proof. In [1], among other things, the following inequality is proved

$$\frac{1}{3} \sqrt{(x + 2y)(2x + y)} \leq \frac{x + y}{2} \quad (x, y \geq 0).$$

Putting $x = F_1, F_2, \ldots, F_n, F_{n+1}, \ldots, F_{2n-1}, F_{2n}$ and $y = F_{2n}, F_{2n-1}, \ldots, F_{n+1}, F_n, \ldots, F_2, F_1$ in the inequality (10), adding and by means of (1) we obtain the stated inequality.

Theorem 3. *The following inequalities*

$$F_n^m F_{n-2} + F_{n-1}^m F_{n-3} + \cdots + F_3^m F_1 \geq \frac{1}{m+1} (F_n^{m+1} - 1), \tag{11}$$

$$F_n^m F_{n-1} + F_{n-1}^m F_{n-2} + \cdots + F_3^m F_2 \leq \frac{1}{m+1} (F_{n+1}^{m+1} - 1) - 1 \tag{12}$$

hold where $m = 0, 1, \ldots$ *and* $n = 1, 2, \ldots$.

Proof. We need the following inequalities

$$(m + 1)y^m(x - y) \leq x^{m+1} - y^{m+1} \leq (m + 1)x^m(x - y)$$

where $x \geq y > 0$ and $m = 0, 1, \ldots$.

Putting $x = F_n, F_{n-1}, \ldots, F_2, F_1$ and $y = F_{n-1}, F_{n-2}, \ldots, F_1, 1$ in the second inequality of (13) and adding we obtain the inequality (11).

Putting $x = F_{n+1}, F_n, \ldots, F_2, F_1$ and $y = F_n, F_{n-1}, \ldots, F_1, 1$ in the first inequality of (13) and adding we obtain the inequality (12).

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ПОЗНАТИ НЕРАВЕНСТВА ЗА КОНЕЧНИ СУМИ СО ФИБОНАЧИЕВИ БРОЕВИ

Mao Qi Ji *, Zdravko F. Starc **

Р е з и м е

Во оваа работа се дадени некои неравенства за конечни суми со Фибоначиеви броеви.

* Department of Mathematics
Suzhou Educational Institute
Suzhou 215002
The People's Republic of China

** Žarka Zrenjanina 93
26300 Vršac
Yougoslavia