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QUASITRIANGULARITY OF THE WEIGHTED SHIFT OPERATORS WITH SPECIAL OPERATOR WEIGHTS

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Abstract

In this paper are given necessary and sufficient conditions for quasitriangularity of certain weighted shift operators with operator weights, in terms of the weight sequence, approximate point spectrum, essential spectrum and Weyl's spectrum. The main result is Theorem 2, where it is shown that if T is a unilateral weighted shift with weights $A_i, i \in \mathbf{N}$ such that $A_i/\|A_i\|$ are isometry, then

T is quasitriangular if and only if $\liminf_{i\in\mathbf{N}} \|Ai\| = 0$

Introduction

Let H be a complex, separable Hilbert space and let $L(H)$ denote the algebra of all bounded, linear operators acting in H. An operator T in $L(H)$ is said to be quasidiagonal (quasitriangular, respectively) if there exists a sequence $(P_n)_{n\in\mathbf{N}}$ of orthogonal projections of finite rank such that $P_n \to 1$ (strongly, $n \to \infty$) and $\|TP_n - P_nT\| \to 0$ as $n \to \infty(\|TP_n - P_nTP_n\| \to 0$ as $n \to \infty)$. It is well known that every quasidiagonal operator is the sum of a bloc-diagonal and compact operator and that every quasitriangular operator is the sum of a triangular and compact operator. The class of quasidiagonal operators is denoted by (QD), whereas the class of quasitriangular operators is denoted by (QT). From the definition of quasidiagonal (quasitriangular) operator, it is easy to see that every quasidiagonal operator is quasitriangular. An operator is said to be biquasitriangular (of

(BQT)) class) if both T and T^* are quasitriangular.

Recall that an operator $T \in L(H)$ has a circular symmetry if T is unitarily equivalent to the $e^{i\theta}T$, $\theta \in R$. Such example of operators are weighted shifts with scalar weights and weighted shift operators with operator weights.

R.A.Smucker, extensively has analyzed quasidiagonal operators in [10] and in [11] he gave a necessary and sufficient conditions for quasidiagonality of weighted shifts

Our purpose in this note is to give some analogical results for quasitriangularity of unilateral weighted shift operators with special operator weights. Firstly, we need some consequences for approximate point spectrum of these operators.

1. Approximate point spectrum of weighted shift operator with special operator weights

Let H_i be Hilbert spaces and $H_i \cong X$, where X is a separable Hilbert space of finite or infinite dimension and let $H = \overset{\infty}{\underset{i=1}{\oplus}} H_i \left(H = \overset{\infty}{\underset{i=-\infty}{\oplus}} H_i\right)$. Let $x = (x_i)$, $y = (y_i) \in H$. It is easy to see that the inner product $(,)$ in H is defined by the equality

$$(x, y) = \sum (x_i, y_i).$$

Where (x_i, y_i) is an inner product of vectors $x_i, y_i \in H_i$ for $i \in \mathbf{N}(i \in \mathbf{Z})$. Next, let $(A_i)_{i \in \mathbf{N}(i \in \mathbf{Z})}$ be a uniformly bounded sequence of operators acting from H_i into H_{i+1}. An operator $T \in L\left(\overset{\infty}{\underset{i=1}{\oplus}} H_i\right)$ is said to be a unilateral weighted shift operator with operator weights A_i if

$$T(x_1, x_2, \ldots, x_n, \ldots) = (0, A_1x_1, A_2x_2, \ldots, A_nx_n, \ldots)$$

for all $x = (x_i) \in \overset{\infty}{\underset{i=1}{\oplus}} H_i$.

An operator $T \in L\left(\overset{\infty}{\underset{i=-\infty}{\oplus}} H_i\right)$ is said to be a bilateral weighted shift operator with operator weights A_i if

$$T(\ldots, x_{-2}, x_1, x_0, x_1, x_2, \ldots) =$$
$$= (\ldots, A_{-3}x_{-3}, A_{-2}x_{-2}, A_{-1}x_{-1}, A_0x_0, A_1x_1, \ldots)$$

for all $x = (x_i) \in \bigoplus_{i=-\infty}^{+\infty} H_i$.

Lemma 1. $T \in L\Big(\bigoplus_{i=1}^{\infty} H_i\Big)$ be a unilateral weighted shift operator with operator weights A_i such that

$$\|A_i x\| = \|A_i\| \, \|x\|, \;\; x \in H_i, \;\; i \in \mathbf{N},$$

then

$$\inf_{\|x\|=1} \|T^n x\|^{1/n} = \inf_{k\in\mathbf{N}} \|A_{k+n-1} \dots A_{k+1} A_k\|^{1/n}.$$

Proof. Obviously,

$$\begin{aligned} \inf_{\|x\|=1} \|T^n x\|^2 &\le \inf_{\|x_k\|=1} \|T^n(0, \dots, 0, x_k, 0, \dots)\|^2 \\ &= \inf_{\|x_k\|=1} \|A_{k+n-1} \dots A_{k+1} A_k x_k\|^2. \end{aligned} \tag{1}$$

Hence, we have

$$\inf_{\|x\|=1} \|T^n x\| \le \inf_{k\in\mathbf{N}} \|A_{k+n-1} \dots A_{k+1} A_k\| \,. \tag{2}$$

On the other hand

$$\begin{aligned} \|T^n x\|^2 &= \sum_{k=1}^{\infty} \|A_{k+n-1} \dots A_{k+1} A_k x_k\|^2 = \\ &= \sum_{k=1}^{\infty} \|A_{k+n-1} \dots A_{k+1} A_k\|^2 \|x_k\|^2 \ge \\ &\ge \inf_{k\in\mathbf{N}} \|A_{k+n-1} \dots A_{k+1} A_k\|^2 \sum_{k=1}^{\infty} \|x_k\|^2 . \end{aligned}$$

Therefore

$$\inf_{\|x\|=1} \|T^n x\|^2 \ge \inf_{k\in\mathbf{N}} \|A_{k+n-1} \dots A_{k+1} A_k\|^2$$

or

$$\inf_{\|x\|=1} \|T^n x\| \geq \inf_{k\in\mathbf{N}} \|A_{k+n-1}\dots A_{k+1}A_k\| . \tag{3}$$

From inequalities (2) and (3) we obtain

$$\inf_{\|x\|=1} \|T^n x\|^{1/n} = \inf_{k\in\mathbf{N}} \|A_{k+n-1}\dots A_{k+1}A_k\|^{1/n}$$

and the proof of the lemma is completed.

Further on, let $\sigma(T)$ denote the spectrum of the operator T, $\sigma_a(T)$ its approximate point spectrum, $\sigma_p(T)$ its point spectrum and $m(T) = \inf_{\|x\|=1} \|Tx\|$ its lower bound. Let $i(T)$ denote $\sup_n m(T^n)^{1/n}$ then it is well known that $i(T) = \sup_n m(T^n)^{1/n} = \lim_{n\to\infty} m(T^n)^{1/n}$ and $\sigma_a(T) \subset \{\lambda \in C: i(T) \leq |\lambda| \leq r(T)\}$, where $r(T)$ is the spectral radius (see [6], [8]). According to Lemma 1 we obtain

$$i(T) = \lim_{n\to\infty} \inf_{\|x\|=1} \|T^n x\|^{1/n} = \lim_{n\to\infty} \inf_{k\in\mathbf{N}} \|A_{k+n-1}\dots A_{k+1}A_k\|^{1/n}. \tag{4}$$

It is easy to see that the spectral radius of the operator T is

$$r(T) = \lim_{n\to\infty} \left(\sup_{k\in\mathbf{N}} \|A_{k+n}A_{k+n-1}\dots A_k\| \right)^{1/2}. \tag{5}$$

The proof of the following proposition is similar to that of Theorem 1 in [8].

Proposition 1. Let

$$T \in L\Big(\overset{\infty}{\underset{i=1}{\oplus}} H_I \Big),\ T(x_i) = (0, A_1x_1, A_2x_2, \dots, A_nx_n, \dots)$$

be a unilateral weighted shift with weights (A_i). Suppose that

$$\|A_i x_i\| = \|A_i\|\,\|x_i\|, \quad i \in \mathbf{N}.$$

Then the approximate point spectrum $\sigma_a(T)$ of the operator T is

$$\sigma_a(T) = \{\lambda \in C: i(T) \leq |\lambda| \leq r(T)\} .$$

Proof. If $i(T) = r(T)$, then since the approximate point spectrum is nonempty and has circular symmetry, we have

$$\sigma_a(T) = \{\lambda \in C : |\lambda| = r(T)\}.$$

Suppose $i(T) < \lambda < r(T)$. Since $\sigma_a(T)$ is closed and has circular symmetry it is sufficient to show that $\lambda \in \sigma_a(T)$. Choose numbers a, b such that $i(T) < a < \lambda < b < r(T)$ and let $\varepsilon > 0$. By (5) we can choose n, k such that $(\lambda/b)^n < \varepsilon$ and $\|A_{k+n-1} \dots A_{k+1}A_k\|^{1/n} > b$. By (4) we can choose p and $m > n + k$ such that $(a/\lambda)^p < \varepsilon$ and $\|A_{m+p} \dots A_{m+1}\|^{1/p} < a$.

Now, we will define the vector $x = (x_i)$ as follows $\|x_k\| = 1$,

$$x_r = \frac{A_{r-1} \dots A_k}{\lambda^{r-k}} x_k, \quad k+1 \le r \le m+p+1$$

$$x_r = 0, \quad r < k \quad \text{and} \quad r > m+p+1.$$

Then

$$(T - \lambda I)x = (A_{i-1}x_{i-1} - \lambda x_i)_{i \in \mathbf{N}},$$

where

$$A_{i-1}x_{i-1} - \lambda x_i = A_{i-1}\frac{A_{i-2} \dots A_k}{\lambda^{i-1-k}} x_k - \lambda \frac{A_{i-1}A_{i-2} \dots A_k}{\lambda^{i-k}} x_k = 0,$$

for $k+1 \le i \le m+p+1$. Therefore

$$Tx - \lambda x = -\lambda x_k - A_{m+p+1}x_{m+p+1}.$$

Further on, since x_k and $A_{m+p+1}x_{m+p+1}$ are orthogonal vectors ($A_{m+p+1}x_{m+p+1} \in H_{m+p+2}$) and since $i(T) < \lambda < r(T)$ we will have

$$\begin{aligned}\|Tx - \lambda x\|^2 &= (Tx - \lambda x, Tx - \lambda x) = \lambda^2\|x_k\|^2 + \|A_{m+p+1}x_{m+p+1}\|^2 \\ &= \lambda^2\|x_k\|^2 + \|A_{m+p+1}\|^2\|x_{m+p+1}\|^2 \\ &\le \|T\|^2(1 + \|x_{m+p+1}\|^2).\end{aligned}$$

Also,

$$\|x\|^2 = \sum_{i=1}^{\infty} \|x_i\|^2 \ge \|x_{k+n}\|^2 + \|x_{m+1}\|^2.$$

But,

$$\begin{aligned}\|x_{k+n}\| &= \left\|\frac{A_{k+n-1} \dots A_k x_k}{\lambda^n}\right\| = \frac{\|A_{k+n-1} \dots A_k\| \, \|x_k\|}{\lambda^n} \\ &= \frac{\|A_{k+n-1} \dots A_k\|}{\lambda^n} \ge \left(\frac{b}{\lambda}\right)^n > \frac{1}{\varepsilon}\end{aligned}$$

and

$$\frac{\|x_{m+p+1}\|}{\|x_{m+1}\|} = \frac{\|A_{m+p}\dots A_{m+1}A_m\dots A_k x_k\|/\lambda^{m+p-k+1}}{\|A_m\dots A_k x_k\|/\lambda^{m-k+1}}$$

$$= \frac{\|A_{m+p}\dots A_{m+1}\|\,\|A_m\dots A_k x_k\|/\lambda^{m+p-k+1}}{\|A_m\dots A_k x_k\|/\lambda^{m-k+1}}$$

$$= \frac{\|A_{m+p}\dots A_{m+1}\|}{\lambda^p} \le \left(\frac{a}{\lambda}\right)^p < \varepsilon\,.$$

Therefore

$$\frac{\|Tx-\lambda x\|^2}{\|x\|^2} \le \|T\|^2 \frac{1+\|x_{m+p+1}\|^2}{\|x_{k+n}\|^2+\|x_{m+1}\|^2}$$

$$\le \|T\|^2 \max\left(\frac{1}{\|x_{k+n}\|^2}, \frac{\|x_{m+p+1}\|^2}{\|x_{m+1}\|^2}\right) < \varepsilon\|T\|^2.$$

From the previous inequality we conclude that $\lambda \in \sigma_a(T)$ and since the approximate point spectrum is closed, it follows that

$$\sigma_a(T) = \{\lambda \in C\colon i(T) \le |\lambda| \le r(T)\}.$$

Proposition 2. Let T be a unilateral weighted shift operator, with operator weights, defined as in Lemma 1. Then the spectrum of T will be

$$\sigma(T) = \{\lambda \in C\colon |\lambda| \le r(T)\}.$$

Proof. Firstly, it is obvious that

$$r(T) = r(T^*) = \limsup_{n\quad k} \|A_k^* A_{k+1}^* \dots A_{k+n-1}^*\|^{1/n}$$

where T^* is the conjugate operator of T.

Let $0 < \lambda < b < r(T)$. We will show that $\lambda \in \sigma_a(T)$. Given $\varepsilon > 0$, we choose n such that $(\lambda/b)^n < \varepsilon$ and k such that

$$\|A_k^* \dots A_{k+n-1}^*\|^{1/n} > b\,.$$

Now, we define the unit vector $x = (xi)_{i\in\mathbf{N}}$, as follows:

$$x_{n+k} \in \overline{R(A_{k+n-1}A_{k+n-2}\dots A_k)} \quad \text{such that} \quad \|x_{n+k}\| = 1,$$

$$x_i = \frac{A_i^* A_{i+1}^* \dots A_{k+n-1}^*}{\lambda^{-i+n+k}}\, x_{n+k}\,, \quad \text{if} \quad 1 \le i \le n+k-1,$$

$$x_i = 0 \quad \text{if} \quad i \ge n+k+1.$$

If $1 \leq i < n+k-1$, then

$$A_i^* x_{i+1} - \lambda x_i = \frac{A_i^* A_{i+1}^* \dots A_{k+n-1}^*}{\lambda^{-i+n+k-1}} x_{n+k} - \lambda \frac{A_i^* \dots A_{k+n-1}^*}{\lambda^{-i+n+k}} x_{n+k} = 0.$$

If $i = n+k-1$, then

$$A_i^* x_{n+k} - \lambda x_{n+k-1} = A_{n+k-1}^* x_{n+k} - \lambda A_{n+k-1}^* \frac{x_{n+k}}{\lambda} = 0.$$

From the above relations we obtain

$$T^*(x_i) - \lambda(x_i) = -\lambda x_{n+k}$$

and therefore

$$\|T^*(x_i) - \lambda(x_I)\| = \lambda. \tag{6}$$

On the other hand,

$$\|x\| \geq \|x_k\| = \left\| \frac{A_k^* A_{k+1}^* \dots A_{k+n-1}^* x_{n+k}}{\lambda^n} \right\| > \left(\frac{b}{\lambda}\right)^n > \frac{1}{\varepsilon}. \tag{7}$$

So by (6) and (7) we have

$$\frac{\|T^* x - \lambda x\|}{\|x\|} = \frac{\lambda}{\|x\|} < \lambda \varepsilon,$$

therefore $\lambda \in \sigma_a(T^*) \subset \sigma(T^*)$. From the circular symmetry of the spectrum it follows that

$$\sigma(T) = \sigma(T^*) = \{\lambda \in C \colon |\lambda| \leq r(T)\}$$

2. Quasitriangular weighted shift operators

By the following theorem we will show the necessary and sufficient conditions for quasitriangularity of the circular operators in terms of the approximate point spectrum.

Theorem 1. Let T be an operator with the circular symmetry and let $\sigma_p(T) \cap \sigma_p(T^*) = \emptyset$. Then $T \in (QT)$ if and only if $\sigma(T) = \sigma_a(T)$.

Proof. If T is quasitriangular operator then $\sigma(T) = \sigma_a(T)$ (see Theorem 1 in [1]). Conversely, assume that $\sigma(T) = \sigma_a(T)$ and $T \notin (QT)$. Then by [2, Corollary. 5.5] there exists a complex number λ such that $T - \lambda I$ is semi-Fredholm and $ind(T - \lambda I) < 0$. It means that

$$\dim Ker(T-\lambda I)-\dim Ker(T-\lambda I)^*<0$$

or

$$\dim Ker(T-\lambda I)-\dim Ker(T^*-\overline{\lambda}I)<0,$$

it implies $\overline{\lambda}\in\sigma_p(T^*)$. Since the operator T has a circular symmetry about the spectrum, $\lambda\in\sigma_p(T^*)$ as well. Now, since $\sigma_p(T)\cap\sigma_p(T^*)=\emptyset$ we have $\lambda\notin\sigma_p(T)$. On the other hand, since $T-\lambda I$ is semi-Fredholm operator, $\overline{R(T-\lambda I)}=R(T-\lambda I)$ and therefore $\|(T-\lambda I)x\|\geq m\|x\|$ for all $x\in H$. This implies $\lambda\notin\sigma_a(T)$ and since $\lambda\in\sigma_p(T^*)$ we have $\lambda\notin\sigma_a(T)$ and $\lambda\in\sigma(T^*)=\sigma(T)$, which contradicts our assumption and the proof of the theorem is completed.

Corollary 1. Let T be an operator with the circular symmetry and let $\sigma_p(T)\cap\sigma_p(T^*)=\emptyset$. Then $T\in(BQT)$ if and only if $\sigma_a(T)=\sigma_a(T^*)$.

Proof. Let $T\in(BQT)$. Then $T\in(QT)$ and $T^*\in(QT)$. From Theorem 1 $\sigma(T)=\sigma_a(T)$ and $\sigma(T^*)=\sigma_a(T^*)$. Therefore $\sigma_a(T)=\sigma_a(T^*)$.

Conversely. Let $\sigma_a(T)=\sigma_a(T^*)$ and suppose that $T\notin(BQT)$. It means that $T\notin(QT)$ or $T^*\notin(QT)$. If $T\notin(QT)$ then $\sigma(T)\neq\sigma_a(T)$. Therefore there exists $\lambda\in\sigma_r(T)\setminus\sigma_a(T)$, where $\sigma_r(T)$ is the residual spectrum of the operator T. Hence $\lambda\in\sigma_p(T^*)\subseteq\sigma_a(T^*)$. Therefore $\sigma_a(T)\neq\sigma_a(T^*)$ and this contradicts the assumption. In case $T^*\notin(QT)$ the contradiction will be obtained analogically.

Corollary 2. Let $T\in L\left(\overset{\infty}{\underset{i=1}{\oplus}}H_i\right)$ be a unilateral weighted shift operator with weighted sequence $\{A_i\}_{i\in\mathbf{N}}$ and let A_i be invertible operators. Then $T\in(QT)$ if and only if $\sigma(T)=\sigma_a(T)$.

Proof. Since A_i are invertible operators, it follows that $\sigma_p(T)=\emptyset$. Hence $\sigma_p(T)\cap\sigma_p(T^*)=\emptyset$. Now, the result immediately follows from Theorem 1.

Example 1. Let $T\in L\left(\overset{\infty}{\underset{i=1}{\oplus}}H_I\right)$ be given by

$$T(x_i)_{i\in\mathbf{N}}=(U_{i-1}x_{i-1})_{i\in\mathbf{N}},U_0x_0=0,\tag{8}$$

where U_i are unitary operators. We will show that

$$\sigma_p(T^*)=\{\lambda\in C:0\leq|\lambda|<1\}\quad\text{and}\quad T\notin(QT).$$

Using relation (8) we have $T^*(x_i)_{i\in\mathbf{N}}=(U_i^*x_{i+1})_{i\in\mathbf{N}}$. Let $\lambda\in\sigma_p(T^*)$. Then there exists a vector $x=(x_i)_{i\in\mathbf{N}}\in\overset{\infty}{\underset{i=1}{\oplus}}H_i$, such that $T^*x=\lambda x$.

Further,

$$(U_i^* x_{i+1})_{i\in \mathbf{N}} = \lambda (x_i)_{i\in \mathbf{N}}$$

or

$$U^* x_{i+1} = \lambda x_i, \quad i \in \mathbf{N}.$$

Consequently,

$$x_2 = \lambda U_1 x_1,$$

$$x_3 = \lambda U_2 x_2 = \lambda^2 U_2 U_1 x_1,$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$x_n = \lambda^n U_{n-1} \dots U_2 U_1 x_1$$

$$\dots \quad \dots \quad \dots \quad \dots$$

Therefore

$$\|x\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 = \|x_1\|^2 + \sum_{n=2}^{\infty} \|\lambda^n U_{n-1} U_{n-2} \dots U_1 x_1\|^2$$

$$= \sum_{n=0}^{\infty} |\lambda|^n \|x_1\|^2,$$

i.e. for $0 \leq |\lambda| < 1$, $\sigma_p(T^*) - \{\lambda \in C: 0 \leq |\lambda| \leq 1\}$. Since $r(T) = r(T^*) = 1$, the spectrum of the operator T is $\sigma(T) = \{\lambda \in C: 0 \leq |\lambda| \leq 1\}$. On the other hand, from Proposition 1 the approximate point spectrum of the operator T is $\sigma_a(T) = \{\lambda \in C: |\lambda| = 1\}$ (because $i(T) = r(T) = 1$). Thus, $\sigma_a(T) \neq \sigma(T)$ and using Corollary 2 we obtain $T \notin (QT)$.

Lemma 2. Let $T \in L\Big(\overset{\infty}{\underset{i=1}{\oplus}} H_i\Big)\Big(T \in L\Big(\overset{\infty}{\underset{i=-\infty}{\oplus}} H_i\Big)\Big)$ be a unilateral (bilateral) weighted shift, with the weighted sequence $\{A_i\}$, and let A_i be invertible operators.

i) If $\dim H_i = n$, $I \in \mathbf{N}$ $(i \in \mathbf{Z})$ then $\dim Ker(T - \lambda I) \leq \dim H_i = n$.

ii) $\dim Ker(T - \lambda I) = \dim Ker(T - \overline{\lambda} I)$,
$\dim Ker(T^* - \lambda I) = \dim Ker(T^* - \overline{\lambda} I)$, for each $\lambda \in C$.

iii) If $\|A_i x\| = \|A_i\| \, \|x\|$ for all $i \in \mathbf{Z}$ and $x \in H_i$, then $\sigma_p(T) \cap \sigma_p(T^*) = \emptyset$.

Proof. i) Let T be a bilateral weighted shift and let $\lambda \in \sigma_p(T)$. Then, there exists a vector $x = (x_i) \in L\Big(\overset{\infty}{\underset{i=-\infty}{\oplus}} H_i\Big)$ such that $Tx = \lambda x$. It follows that $(A_{i-1}x_{i-1}) = (\lambda x_i)$ and calculations show that the vector x is of the form:

$$x = \Big(\dots, \lambda^2 A_{-2}^{-1}A_{-1}^{-1}x_0, \lambda A_{-1}^{-1}x_0, x_0, \frac{1}{\lambda}A_0x_0, \frac{1}{\lambda^2}A_1A_0x_0, \dots\Big) \quad (9)$$

where $x_0 \in H_0$. Further, let $x_0^1, x_0^2, \dots, x_0^p$ linearly independent p–vectors in H_0 that occur on the 0-position of eigenvectors. Now, the corresponding eigenvectors will be

$$x_i = \Big(\dots, \lambda^2 A_{-2}^{-1}A_{-1}^{-1}x_0^i, \lambda A_{-1}^{-1}x_0^i, x_0^i, \frac{1}{\lambda}A_0x_0^i, \frac{1}{\lambda^2}A_1A_0x_0^i, \dots\Big),$$

$i = 1, 2, \dots, p$.. Every other eigenvector $y = (y_i) \in \bigoplus_{i=-\infty}^{\infty} H_i$ is of the form

$$y = \Big(\dots, \lambda^2 A_{-2}^{-1}A_{-1}^{-1}y_0, \lambda A_{-1}^{-1}y_0, y_0, \frac{1}{\lambda}A_0y_0, \frac{1}{\lambda^2}A_1A_0y_0, \dots\Big).$$

Since $y_0 \in H_0$, there exists complex numbers $\alpha_1, \alpha_2, \dots \alpha_p$ so that $y_0 = \alpha_1 x_0^1 + \alpha_2 x_0^2 + \dots + \alpha_p x_0^p$. Therefore, $y = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_p x_p$. This implies that $\dim Ker(T - \lambda I) \leq \dim H_0 = n$.

ii) If x is eigenvector corresponding to the eigenvalue λ, then it is expressed by formula (9). Then the vector $\overline{x}$ defined by the formula

$$\overline{x} = \Big(\dots, \overline{\lambda}^2 A_{-2}^{-1}A_{-1}^{-1}x_0, \overline{\lambda} A_{-1}^{-1}x_0, x_0, \frac{1}{\overline{\lambda}}A_0x_0, \frac{1}{\overline{\lambda}^2}A_1A_0x_0, \dots\Big),$$

is an eigenvector corresponding to the eigenvalue $\overline{\lambda}$ and conversely. By the definition of the vector $\overline{x}$ it is obvious that $\overline{x} \in \bigoplus_{i=-\infty}^{\infty} H_i$ and therefore $\overline{\lambda} \in \sigma_p(T)$. Since the map $x \to \overline{x}$ is one to one and onto, it follows that $\dim Ker(T - \lambda I) = \dim Ker(T - \overline{\lambda} I)$. Analogously, it can be shown that $\dim Ker(T^* - \lambda I) = \dim Ker(T^* - \overline{\lambda} I)$, $\lambda \in C$. From the equation $T^*y = \lambda y$, we obtain that the vector $y = (y_i) \in \bigoplus_{i=-\infty}^{\infty} H_i$ is of the form:

$$y = \Big(\dots, \frac{1}{\lambda^2}A_{-2}^*A_{-1}^*y_0, \frac{1}{\lambda}A_{-1}^*y_0, y_0, \lambda A_0^{*\,-1}, y_0, \lambda^2 A_1^{*\,-1}A_0^{*\,-1}y_0, \dots\Big). \quad (10)$$

Then the vector

$$\overline{y} = \Big(\dots, \frac{1}{\lambda^2}A_{-2}^*A_{-1}^*y_0, \frac{1}{\overline{\lambda}}A_{-1}^*y_0, y_0, \overline{\lambda} A_0^{*\,-1}y_0, \overline{\lambda}^2 A_1^{*\,-1}A_0^{*\,-1}y_0, \dots\Big)$$

is eigenvector corresponding to the eigenvalue $\overline{\lambda}$ (because $T^*\overline{y} = \overline{\lambda}y$). Also, the correspondence $y \to \overline{y}$ is one to one and onto. Hence $\dim Ker(T^* - \lambda I) = \dim Ker(T^* - \overline{\lambda} I)$.

iii) Let $\sigma_p(T) \neq \emptyset$ and $\sigma_p(T^*) \neq \emptyset$. Assume that $\sigma_p(T) \cap \sigma_p(T^*) \neq \emptyset$ and let $\lambda \in \sigma_p(T) \cap \sigma_p(T^*)$. Then there exists vectors $x, y \in \bigoplus_{i=-\infty}^{\infty} H_i$ such that $Tx = \lambda x$ and $T^*y = \lambda y$. Further on, the vector x is expressed by formula (9) and the vector y by formula (10). So

$$\|x\| = \left(\cdots + \|\lambda^2 A_{-2}^{-1} A_{-1}^{-1}\|^2 + \|\lambda A_{-1}^{-1}\|^2 + 1 + \left\|\frac{1}{\lambda} A_0\right\|^2 + \cdots\right)^{1/2} \|x_0\| < \infty.$$

Then

$$x'\left(\ldots, \lambda^2 A_{-2}^{-1} A_{-1}^{-1} e, \lambda A_{-1}^{-1} e, e, \frac{1}{\lambda} A_0 e, \frac{1}{\lambda^2} A_1 A_0 e, \ldots\right) \qquad (11)$$

is eigenvector of the operator T (i.e. $Tx' = \lambda x'$ and $\|x'\| = \|x\|/\|x_0\| < \infty$) for every basic vector e in H_0. Analogously, we conclude that

$$y' = \left(\ldots, \frac{1}{\lambda^2} A_{-2}^* A_{-1}^* e, \frac{1}{\lambda} A_{-1}^* e, e, \lambda A_0^{*\,-1} e, \lambda^2 A_1^{*\,-1} A_0^{*-1} e. \ldots\right), \qquad (12)$$

is eigenvector of the operator T^* corresponding to the eigenvalue λ. Therefore,

$$(Tx', T^*y') = (\lambda x', \lambda y') = |\lambda|^2 (x', y').$$

On the other hand

$$(Tx', T^*x') = (T^2 x', y') = \lambda^2 (x', y').$$

From the last equality, it follows that $|\lambda|^2 (x', y') = \lambda^2 (x', y')$. This is possible if and only if $(x', y') = 0$. Now, by (11) and (12) we obtain

$$(x', y') = (e, e) + (e, e) + \cdots + (e, e_+ \cdots = 0.$$

Hence $(e, e) = \|e\|^2 = 0$ and this is impossible since e is a unit vector. The proof of the Lemma 2 in case T is a unilateral weighted shift is trivial and we omit it.

Remark 1. From Lemma 2 if λ is replaced with $e^{i\theta}\lambda (0 \leq \theta \leq 2\pi)$ then the corresponding eigenvectors have the following form:

$$y = (\ldots, (e^{i\theta}\lambda)^2 A_{-2}^{-1} A_{-1}^{-1} x_0, (e^{i\theta}\lambda) A_{-1}^{-1} x_0, x_0, \frac{1}{e^{i\theta}\lambda} A_0 x_0, \frac{1}{(e^{i\theta}\lambda)^2} A_1 A_0 x_0, \ldots).$$

It means that $\dim Ker(T - \lambda I)$ =const. for $\lambda\, e^{i\theta}$. Analogously, we conclude that $\dim Ker(T^* - \lambda I)$ =const. for $\lambda = \lambda\, e^{i\theta}$.

Remark 2. Because the Weyl's spectrum $W(T)$ of the operator T, contains all points of the spectrum $\sigma(T)$ except eigenvalues of finite multiplicity, $W(T+K)$ has a circular symmetry for every compact operator K. The essential spectrum $\sigma(\Pi(T)) = W(T) \setminus \{\lambda \in C: \lambda I - T \notin F_0(H)\}$ has a circular symmetry also, where $F_0(H)$ is the set of Fredholm operators with index equal to zero.

Theorem 2. Let $T \in L\Big(\overset{\infty}{\underset{i=1}{\oplus}} H_i\Big)\Big(T \in L\Big(\overset{\infty}{\underset{i=-\infty}{\oplus}} H_i\Big)\Big)$ be a unilateral (bilateral) weighted shift with the weighted sequence $\{A_i\}_{i\in\mathbf{N}}(\{A_i\}_{i\in\mathbf{Z}})$.

i) If $A_i, i \in \mathbf{N}(i \in \mathbf{Z})$ are invertible operators and $\dim H_i = n$ then $T \in (QT)$ if and only if $\sigma(\Pi(T)) = W(T)$.

ii) If $T \in L\Big(\overset{\infty}{\underset{i=1}{\oplus}} H_i\Big)$ and $\|A_i x\| = \|A_i\|\,\|x\|$, $x \in H_i$, $i \in \mathbf{N}$ then

$T \in (QT)$ if and only if $\liminf_{i\in\mathbf{N}} \|A_i\| = 0$.

Proof. i) Let $T \in (QT)$. Since for the weighted shifts is always $T^* \in (QT)$, it follows that $T \in (BQT)$ which implies $\sigma(\Pi(T)) = W(T)$ (see [7]).

Conversely, assume that

$$\sigma(\Pi(T)) = W(T) \quad \text{and} \quad T \notin (QT).$$

Then, by Theorem 1 in [2] there exists $\lambda \in C$ such that $T - \lambda I$ is semi-Fredholm with $ind(T - \lambda I) < 0$. It means that

$$\overline{R(T-\lambda I)} = R(T-\lambda I)$$

and

$$\dim Ker(T-\lambda I) < \infty \quad \text{or} \quad \dim Ker(T^* - \overline{\lambda} I) < \infty\,.$$

Now, from Lemma 2 we have

$$(T-\lambda I) \in F(H) \quad \text{and} \quad ind(T-\lambda I) < 0\,.$$

Therefore,

$$\lambda \in W(T) \quad \text{and} \quad \lambda \notin \sigma(\Pi(T))\,.$$

Hence,

$$\sigma(\Pi(T)) \neq W(T)\,,$$

which contradicts assumption.

ii) If T is a unilateral weighted shift operator, then by Corollary 2 $T \in (QT)$ if and only if $\sigma(T) = \sigma_a(T)$. Now, by Proposition 1 and Proposition 2 $\sigma(T) = \sigma_a(T)$ if and only if $i(T) = \lim_n \inf_k \|A_{k+n-1} \dots A_{k+1}A_k\|^{1/n} = 0$

and this is possible if and only if $\liminf_{i \in \mathbf{N}} \|A_i\| = 0$.

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КВАЗИТРИАНГУЛАРНОСТ НА ТЕЖИНСКИ ШИФТ ОПЕРАТОРИ СО СПЕЦИЈАЛНИ ОПЕРАТОРСКИ ТЕЖИНИ

Muhib R. Lohaj

Р е з и м е

Во оваа работа се дадени потребни и доволни услови за квазитриангуларност на некои тежински шифт оператори со операторска тежина, изразена во форма на тежинска низа, апроксимативен точкест спектар, суштински спектар и спектарот на Weyl. Главниот резултат е Теорема 2, каде е докажано дека ако T е унилатерален тежински шифт со тежини A_i $i \in \mathbf{N}$ така што $A_i/\|A_i\|$ е изометрија, тогаш T е квазитриангуларен ако и само ако

$$\liminf_{i \in N} \|Ai\| = 0\,.$$

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