Математички Билтен
24 (L)
2000 (69–76)
Скопје, Македонија

ISSN 0351–336X

INTRODUCTION OF BLASCHKE DISTRIBUTIONS AND APPROXIMATION OF DISTRIBUTION IN D' BY A SEQUENCE OF FINITE BLASCHKE DISTRIBUTIONS

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Abstract

In this work Blaschke distributions are defined and used to approximate a distribution in D'.

0. Introduction

0.1: Some background on Blaschke products

Let U be the open, unit disc in the plane, $T = \partial U$. We denote by $H^\infty(U)$ the algebra of bounded analytic functions in U. If $f \in H^\infty(U)$, then the radial boundary function

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

is defined almost everywhere on T with respect to the Lebesque measure on T and $\log|f^*(e^{i\theta})| \in L^1(T)$. A function $f \in H^\infty(U)$ is an inner function if $|f| \leq 1$ and $|f^*| = 1$ a.e. with respect to the Lebesque measure on T.

Let $\{z_n\}$ be a sequence of points in U such that

$$\sum_{n=1}^{\infty}(1-|z_n|) < \infty\,. \qquad (0.1.1)$$

Let m be the number of z_n equal to 0. Then the infinite product

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z} \qquad (0.1.2)$$

converges on U. The function $B(z)$ of the form (0.1.2) is caled Blaschke product. $B(z)$ is in $H^\infty(U)$, and the zeros of $B(z)$ are precisely the points z_n, each zero having multiplicity equal the number of times it occurs in the sequence $\{z_n\}$.

Moreover $|B(z)| \leq 1$ and $|B^*(e^{i\theta})| = 1$ a.e. Thus every Blaschke product is an inner function. For the needs of our subsequent work we will define the Blaschke product in the upper half plane Π^+. In the upper half plane Π^+, condition (0.1.1) is replaced by

$$\sum_{n=1}^{\infty} \frac{y_n}{1 + |z_n|^2} < \infty, \quad z_n = x_n + iy_n \in \Pi^+ \cdots \qquad (0.1.3)$$

and the Blaschke product with zeros $\{z_n\}$ is

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{n=1}^{\infty} \frac{|z_n^2 + 1|}{z_n^2 + 1} \frac{z - z_n}{z - \overline{z}_n}. \qquad (0.1.4)$$

Note: If the number of zeros z_n in (0.1.2) or (0.1.4) is finite, then we call $B(z)$ finite Blaschke product.

The following result is given in [3].

Theorem 0.1.1. *Let $f(z)$ be analytic in the open unit disc U and continous in $\overline{U}$. Suppose $0 < |f(z)| \leq 1$ on $|z| = 1$ and let E be the subset of $|z| = 1$ on which $|f(z)| < 1$. Suppose E is nonempty. Then there exists a sequence $\{B_k(z)\}$ of finite Blaschke products with simple zeros, such that $|B_k(z)| \to |f(t)|$ uniformly in each closed subset of $\overline{U} \setminus \overline{E}$, and $B_k(z) \to f(z)$ uniformly in each closed subset of U.*

Now, if $f(z)$ is analytic function in U such that $|f(z)| \leq 1$ on T, then applying T.0.1.1 to the functions $f(rz)$, $0 < r < 1$ and using the fact that $\frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{it})| dt \to 0$, as $r \to 1$ iff h is a Blaschke product, it can be shown that $f(z)$ can be approximated by a sequence of finite Blaschke products $\{B_k(z)\}$ uniformely on each compact subset of U.

We will use the analog result for the upper half plane Π^+ i.e. that every function $f(z)$ analytic in the upper half plane Π^+, $|f(z)| < 1$ on Π^+, $|f^*(x)| \leq 1$, $x \in \mathbf{R}$ can be approximated by a sequence of finite Blaschke products $\{B_k(z)\}$ uniformly on each compact subset of Π^+.

0.2: Some background on distributions and analytic representation of distributions

$D = D(\mathbf{R}^n)$ denote the space of all complex valued infinitely differentiable functions on $\mathbf{R}^n$ which have compact support. Convergence in D is defined in the following way. A sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in D as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset \mathbf{R}^n$ such that $\operatorname{supp}(\varphi_\lambda) \subseteq K$ for each λ, $\operatorname{supp}(\varphi) \subseteq K$ and for every n-tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on K as $\lambda \to \lambda_0$.

$D' = D'(\mathbf{R}^n)$ is the space of all continuons linear functionals on D, where continuity means that $\varphi_\lambda \to \varphi$ in D as $\lambda \to \lambda_0$ implies $\langle T, \varphi_\lambda\rangle \to \langle T, \varphi\rangle$ as $\lambda \to \lambda_0$, $T \in D'$.

$E = E(\mathbf{R}^n)$ denote the space of all complex valued infinitely differentiable functions on $\mathbf{R}^n$. Convergence in E is defined in the following way. A sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in E$ converges to $\varphi \in E$ in E as $\lambda \to \lambda_0$ if and only if for every n-tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on every compact subset K of $\mathbf{R}^n$, as $\lambda \to \lambda_0$.

E' is the space of all continous linear functionals on E.

Note: D' is called the space of distributions, E' is the space of compact distributions.

We say that a sequence $\{T_j\}$ of distributions T_j that are elements of one of the above distribution spaces (D' or E') converges weakly to a distribution T of the same space if

$$\langle T, \varphi\rangle = \lim_{j\to\infty} \langle T_j, \varphi\rangle,$$

for every φ in the apropriate test space (either D or E).

Now, let T be a given distribution in $D'(\mathbf{R})$. Any pair of functions $f_+(z), f_-(z)$ which are defined and analytic in the upper and lower half plane, respectively such that:

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} [f_+(x+i\varepsilon) - f_-(x-i\varepsilon)]\varphi(x)dx = \langle T, \varphi\rangle \qquad (0.2.1)$$

for all $\varphi \in D(\mathbf{R})$ is called an analytic representation of T.

We shall use the terminology "analytic representation" defined above for other spaces of distributions as well as for $D'(\mathbf{R})$ when we represent these distributions as in (0.2.1) for φ in the coresponding function (test) space.

For distributions of the space E' it is known the analytic representation.

Let $T \in E'(\mathbf{R})$. The function

$$C(T;z) = \frac{1}{2\pi i}\langle T_t, \frac{1}{t-z}\rangle$$

for z varying over an appropiate subset of $\mathbf{C}$, is called Cauchy integral of T.

Let $E_B(\mathbf{R})$ denote the subspace of $E(\mathbf{R})$ consisting of all bounded functions in $E(\mathbf{R})$. We now state an analytic representation theorem for distributions in $E'(\mathbf{R})$.

Theorem 0.2.1. *If $T \in E'(\mathbf{R})$, then*

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} [C(T;x+i\varepsilon) - C(T;x-i\varepsilon)]\varphi(x)dx = \langle T, \varphi\rangle$$

for all $\varphi \in E_B(\mathbf{R})$.

A proof can be found in [2].

The construction of on analytic representation of on arbytrary distribution in $D'(\mathbf{R})$ by means of its Cauchy integral is not always possible. Yet we do have analytic representation results for $D'(\mathbf{R})$.

Theorem 0.2.2. *Every distribution $T \in D'(\mathbf{R})$ has an analytic representation.*

For proof, see [1].

1. Definition of a Blaschke distributions

Let $B(z)$ be a Blaschke product, $z = x + iy \in \Pi^+$, with zeros $\{z_n\}$ that belong to the upper half plane Π^+.

With $\langle B^+, \varphi\rangle$ we denote

$$\langle B^+, \varphi\rangle = \lim_{y\to 0+} \int_{-\infty}^{\infty} B(z)\varphi(x)dx\,, \quad z = x+iy \in \Pi^+, \quad \varphi \in D(\mathbf{R})\,. \tag{1.1}$$

Clearly, it is a linear functional on $D(\mathbf{R})$ i.e. for $\varphi_1, \varphi_2 \in D(\mathbf{R})$ and $\lambda_1, \lambda_2 \in \mathbf{C}$, we have

$$\langle B^+, \lambda_1\varphi_1 + \lambda_2\varphi_2\rangle = \lim_{y\to 0+} \int_{-\infty}^{\infty} B(z)[\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x)]dx =$$

$$= \lim_{y\to 0+} \left[\int_{-\infty}^{\infty} B(z)\lambda_1\varphi_1(x)dx + \int_{-\infty}^{\infty} B(z)\lambda_2\varphi_2(x)dx\right] =$$

$$= \lambda_1 \lim_{y\to 0+} \int_{-\infty}^{\infty} B(z)\varphi_1(x)dx + \lambda_2 \lim_{y\to 0+} \int_{-\infty}^{\infty} B(z)\varphi_2(x)dx =$$

$$= \lambda_1\langle B^+, \varphi_1\rangle + \lambda_2\langle B^+, \varphi_2\rangle\,.$$

$\langle B^+, \varphi \rangle$ is continous: let $\{\varphi_\lambda\}$, $\varphi_\lambda \in D(\mathbf{R})$, $\varphi_\lambda \to \varphi$ in $D(\mathbf{R})$ as $\lambda \to \lambda_0$. That means that there is a compact set K such that $\operatorname{supp}(\varphi_\lambda) \subseteq K$, for each λ, $\operatorname{supp}(\varphi) \subseteq K$ and $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on K as $\lambda \to \lambda_0$, for each n-tuple α of nonnegative integers. Then:

$$\begin{aligned}\lim_{\lambda\to\lambda_0} \langle B^+, \varphi_\lambda \rangle &= \lim_{\lambda\to\lambda_0} \lim_{y\to 0^+} \int_{-\infty}^{\infty} B(z)\varphi_\lambda(x)dx = \\ &= \lim_{\lambda\to\lambda_0} \lim_{y\to 0^+} \int_K B(z)\varphi_\lambda(x)dx = \\ &= \lim_{y\to 0^+} \int_K B(z) \lim_{\lambda\to\lambda_0} \varphi_\lambda(x)dx = \\ &= \lim_{y\to 0^+} \int_K B(z)\varphi(x)dx = \\ &= \lim_{y\to 0^+} \int_{-\infty}^{\infty} B(z)\varphi(x)dx = \langle B^+, \varphi \rangle .\end{aligned}$$

In the above computations, we used that

$$\lim_{y\to 0^+} \lim_{\lambda\to\lambda_0} \int_K B(z)\varphi_\lambda(x)dx = \lim_{\lambda\to\lambda_0} \lim_{y\to 0^+} \int_K B(z)\varphi_\lambda(x)dx . \tag{1.2}$$

Now we will give proof of (1.2).

Proof of (1.2).

Note: In the proof instead of $B(z)$ we will write $B(x, y)$.

Let us consider the sequence $\{f_\lambda(y)\}_\lambda$, where

$$f_\lambda(y) = \int_K B(x, y)\varphi_\lambda(x)dx .$$

We have

$$\lim_{\lambda\to\lambda_0} f_\lambda(y) = \lim_{\lambda\to\lambda_0} \int_K B(x, y)\varphi_\lambda(x)dx = \int_K B(x, y)\varphi(x)dx = f(y)$$

(this is possible because $\varphi_\lambda(x)$ converges to $\varphi(x)$ uniformly on K).

Thus, the sequence $\{f_\lambda(y)\}_\lambda$ converges pointwise to $f(y)$, as $\lambda \to \lambda_0$

$$0 \le \sup_{y\in(0,\infty)} |f_\lambda(y) - f(y)| = \sup_y \left| \int_K B(x, y)\varphi_\lambda(x)dx - \int_K B(x, y)\varphi(x)dx \right| \le$$

$$\le \sup_y \int_K |B(x, y)|\,|\varphi_\lambda(x) - \varphi(x)|dx \overset{|B(x,y)|<1}{\le} \int_K |\varphi_\lambda(x) - \varphi(x)|dx .$$

But,
$$\lim_{\lambda\to\lambda_0}\int_K |\varphi_\lambda(x)-\varphi(x)|dx = 0\,,$$
so
$$\lim_{\lambda\to\lambda_0}\sup_{y\in(0,\infty)}|f_\lambda(y)-f(y)| = 0\,,$$
which means that the sequence $f_\lambda(y)$ converges uniformly to $f(y)$ as $\lambda\to\lambda_0$ and thus the change of the limits is possible.

Definition 1.1. $\langle B^+,\varphi\rangle$ for $\varphi\in D$ defined with (1.1), where $B(z)$ is a Blaschke product with zeros in Π^+, is a distribution on D and we name it an **upper Blaschke distribution** on D.

Similarly, for a $B(z)$ being a Blaschke product with zeros z_n, $n\in\mathbf{N}$ that belong to the lower half plane Π^-, with $\langle B^-,\varphi\rangle$, for $\varphi\in D$ we denote

$$\langle B^-,\varphi\rangle = \lim_{y\to 0^+}\int_{-\infty}^{\infty} B(z)\varphi(x)dx\,,\quad z=x-iy\in\Pi^-\,,\quad \varphi\in D(\mathbf{R})\,. \tag{1.3}$$

Definition 1.2. $\langle B^-,\varphi\rangle$ for $\varphi\in D$, defined with (1.3), where $B(z)$ is a Blaschke product with zeros in Π^-, is a distribution on D and we name it an **lower Blaschke distribution** on D.

Note: In the above definitions, when $B(z)$ is finite Blaschke product, we will call the associate Blaschke distribution finite Blaschke distribution.

2. Approximation of distribution by a sequence of finite Blaschke distributions

As the Blaschke product has usefull value in function spaces, so will the Blaschke distribution play an importanat role in the distribution spaces.

Theorem 2.1. *Let T be a distribution on $D(\mathbf{R})$ such that their analityc continuation f_+,f_- satisfy $|f_+(z)|<1$, $z\in\Pi^+$, $|f_+^*(x)|\le 1$, $x\in\mathbf{R}$, $|f_-(z)|<1$, $z\in\Pi^-$, $|f_-^*(x)|\le 1$, $x\in\mathbf{R}$, Then there are sequences of finite upper and lower Blaschke distributions B_k^+,B_k^- such that*

$$\langle T,\varphi\rangle = \lim_{k\to\infty}[\langle B_k^+,\varphi\rangle - \langle B_k^-,\varphi\rangle]\,,\quad \varphi\in D(\mathbf{R})\,.$$

Proof. Let $T\in D'(\mathbf{R})$. Then there are functions $f_+(z),f_-(z)$ that are analytic in the upper and lower half plane, respectively, such that

$$\langle T,\varphi\rangle = \lim_{y\to 0^+}\int_{-\infty}^{\infty}[f_+(x+iy)-f_-(x-iy)]\varphi(x)dx\,,\quad \forall\varphi\in D(\mathbf{R})\,.$$

Let the functions f_+ and f_- satisfy $|f_+(z)| < 1$, $z \in \Pi^+$, $|f_+^*(x)| \leq 1$, $x \in \mathbf{R}$, $|f_-(z)| < 1$, $z \in \Pi^-$, $|f_-^*(x)| \leq 1$, $x \in \mathbf{R}$.

Then, by theorem of approximation there exist sequences of finite Blaschke products $B_k^+(z)$ with simple zeros in Π^+ and $B_k^-(z)$ with simple zeros in Π^- such that $B_k^+(z) \to f_+(z)$, $k \to \infty$ uniformly on each compact subset of Π^+, and $B_k^-(z) \to f_-(z)$, $k \to \infty$ uniformly on each compact subset of Π^-.

Let $\operatorname{supp}(\varphi) = K \subset \mathbf{R}$. Now, we have:

$$\begin{aligned}
\langle T, \varphi\rangle &= \lim_{y\to 0^+} \int_{-\infty}^{\infty} [f_+(x+iy) - f_-(x-iy)]\varphi(x)dx = \\
&= \lim_{y\to 0^+} \int_K \lim_{k\to\infty} B_k^+(x+iy) - \lim_{k\to\infty} B_k^-(x-iy)]\varphi(x)dx = \\
&= \lim_{y\to 0^+} \int_K \lim_{k\to\infty} [B_k^+(x+iy) - B_k^-(x-iy)]\varphi(x)dx = \\
&= \lim_{k\to\infty} \left[\lim_{y\to 0^+} \int_K B_k^+(x+iy)\varphi(x)dx - \lim_{y\to 0^+} \int_K B_k^-(x-iy)\varphi(x)dx\right] = \\
&= \lim_{k\to\infty} \left[\lim_{y\to 0^+} \int_{-\infty}^{\infty} B_k^+(x+iy)\varphi(x)dx - \lim_{y\to 0^+} \int_{-\infty}^{\infty} B_k^-(x-iy)\varphi(x)dx\right] \\
&= \lim_{k\to\infty} [\langle B_k^+, \varphi\rangle - \langle B_k^-, \varphi\rangle].
\end{aligned}$$

Note: Because of similar arguments as in the proof of (1.2), the change of limits in the above computations is possible.

Thanks to prof. Nikola Pandeski for his suggestions for this work.

References

[1] Бремерман, Г.: *Распределения, комплексные переменные и преобразования Фурье*, МИР, Москва (1968).

[2] Carmichael, R. D. and Mitrović, D.: *Distributions and analytic funkctions*, John Wiley and Sons, Inc., New York (1988).

[3] Duren, P. L.: *Theory of H^p Spaces*, Acad. Press, New York (1970).

[4] Garnett, J. *Bounded analytic funktion*, Acad. Press, New York (1981).

[5] Schwartz, L.: *Theorie des distributions*, Tome I, Tome II, Herman and C^ie^, Paris (1950, 1951).

[6] Stanković B. and Pilipović S.: *Teorija distribucija*, PMF, Novi Sad (1988).

ДЕФИНИЦИЈА НА БЛАШКЕОВИ ДИСТРИБУЦИИ И АПРОКСИМАЦИЈА НА ДИСТРИБУЦИИ ВО D' СО НИЗА ОД КОНЕЧНИ БЛАШКЕОВИ ДИСТРИБУЦИИ

Весна Манова Ераковиќ

Резиме

Во оваа работа се дефинираат Блашкеови дистрибуции и воведените Блашкеови дистрибуции се користат за апроксимација на дистрибуции во D',

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