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FINITE PROCESS ALGEBRAS

Aleksandar Nanevski

Abstract

PA_ε algebras are strucures of specific importance for parallel processing. They are the basis for mathematcal representation of the idea of parallel computation through the formalisms of the process algebras.

The number of equations (axioms) that the elements of those algebras have to satisfy increase as their signatures are getting augmented in order to gain more expressive power. However, all of them retain the three basic properties: commutativity, associativity and idempotency of their additive operation. That makes it possible to describe and analyse them as semilattices.

This paper presents some properties of these structures that can simplify, in the case of finite cardinality, their automatic generation.

1. Introduction

The process algebras are mathematical structures that can represent the behavoir of various computational systems and describe the execution traces of algorithms that they implement. They are being used in proving of the correctness of network protocols, systolic arrays and several other kinds of parallel and distributed systems. Because of that, exploration of their properties is of interest for the theory as well as the practice. For its simplification, the databases of process algebras might be describe as a tool for initial testing of the hypotheses. However, generation of these databases is problem of exponential compexity, and can, accordingly, require severe dedication of computational resourses. In the following sections, we describe the generation of databases of PA_ε agebras of three, four and five elements, together with the optimizations that considerably shortened the processing time.

2. Definitions

Definition 1. *A signature is said to be a finite set of* functional symbols (operators). *Every symbol from the signature is characterized by its arity – an integer that denotes the number of symbol's arguments. Symbols of* 0 *arity are called* atomic actions (constants).

In the further text, letters $a, b, c, \dots$ will be used to represent the atomic actions. Besides the operators from the signature, we use another kind od symbols – variables, and we represent them with letters x, y, z, with or without indices.

Definition 2. *A term over signature* Σ *(or a* Σ*-term) is defined in the following way:*

(i) *the atomic actions and the variables are terms,*

(ii) *if* f *is* n*-ary operator and* $t_1, t_2, \dots, t_n$ *are terms, then* $f(t_1, t_2, \dots, t_n)$ *is also a term with* $t_1, t_2, \dots, t_n$ *being its subterms,*

(iii) *every term is derived from the above rules in finitely many steps.*

If f is a binary operator (operator of arity 2), then the term $f(t_1, t_2)$ will be written as $(t_1 f t_2)$. Very often, in the cases when the priority of the operators is explicitly specified, the terms will be writen without unnecessary parentheses.

Terms that do not contain variables, but are consisted solely of non-null operators and atomic actions, are called *closed terms.* If a term contains variables as well, it is called an *open term.*

Definition 3. Specification *is an ordered pair* (Σ, E) *where* Σ *is a signature, and* E *is a set of equations. Equations in* E *are called* axioms, *and are of the form* $t_1 = t_2$ *where* t_1 *and* t_2 *are* Σ*-terms.*

Definition 4. Algera $\mathbf{A}$ *is an ordered pair* (A, F), *where* A *is a set, called* the carrier of $\mathbf{A}$ and F *is a set of functions of the type* $f\colon A^{k_f} \to A$. *Functions* f *are called* operations, *while* k_f *are their* arities.

Definition 5. *The algebra* $\mathbf{A}$ *is a* Σ*-algebra (for some specific signature* Σ*) if there is a bijection between the set of functions of* $\mathbf{A}$ *and the set of operators of* Σ *such that every function is mapped to an operator with the same arity. The bijection is called an* interpreration *of* Σ *in* $\mathbf{A}$.

Given an algebra $\mathbf{A}$ every interpretation can be obviously extended so that to each term t over different variables $x_1, x_2, \dots, x_n$ (ordered as their appear in the term) corresponds an n-ary mapping $\bar{t}$: $A^n \to A$. If p_i, $i = 1, 2, \dots, n$, are elements od A, then $\bar{t}(p_1, p_2, \dots, p_n)$ denotes the value otained when every x_i is replaced by corresponding p_i.

Definition 6. *Let* $\mathbf{A}$ *be a* Σ*-algebra, for some signature* Σ. *The element* $p \in A$ *is called* finite *if it is an interpretation of a closed term. Otherwise,* p *is* infinite.

Definition 7. *Let* Σ *be a signature,* $\mathbf{A}$ *a* Σ*-algebra, and* t_1 *and* t_2, *respectively, an* n*-ary and* m*-ary* Σ*-terms. Given some interpretation of*

Σ *in* $\mathbf{A}$, *the equation* $t_1 = t_2$ holds *in* $\mathbf{A}$ *if*:

$$\forall p_1, p_2, \ldots, p_{m+n} \in A \quad \bar{t}_1(p_1, p_2, \ldots, p_n) = \bar{t}_2(p_{n+1}, p_{n+2}, \ldots, p_m)\,.$$

Then $\mathbf{A}$ *is called a* model *for the equation* $t_1 = t_2$ *and this is denoted by* $\mathbf{A} \models t_1 = t_2$.

Definition 8. *Let* (Σ, E) *be a specification. The* Σ*-algebra* $\mathbf{A}$ *is called a* model *for* E *if it is model for every equation in* E. $\mathbf{A}$ *is referred to as algebra over* (Σ, E), *and this is denoted by* $\mathbf{A} \models E$.

Definition 9. Recursive specification (*on the set* V *of variables*) *over the signature* Σ *is system of equations of the form:*

$$x = s_x(V)$$

where $x \in V$ *and* s_x *are* Σ*-terms with variables from* V. *Usually, one variable is distinguished and is referred to as a* root variable.

Complete solution of *a recursive specification in some* Σ*-algebra* $\mathbf{A}$ *is said to be any mapping* $\varphi: V \to A$ *for which* $\varphi(x) = \bar{s}_x\big(\varphi(V)\big)$, *for all* $x \in V$. *If* x *is the root variable of a recursive specification, then* $\varphi(x)$ *is said to be its* solution.

Definition 10. *Two recursive specifications over* (Σ, E) *are* equivalent *if they have equal solutions in every algebra over* (Σ, E).

Definition 11. (***BPA* algebra**). *Let* Σ_{BPA} *be a signature with two binary operators* $+$ *and* $\cdot$ (*with* $\cdot$ *having higher priority than* $+$), *while* E_{BPA} *is the following set of equations*:

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$
$$x + x = x$$
$$(x + y)z = xz + yz$$
$$(xy)z = x(yz)\,.$$

Every Σ_{BPA}*-algebra, model for* E_{BPA}, *is called a BPA algebra.*

Definition 12. *Let* Σ *be a signature that contains the operators of* Σ_{BPA}. *We say that appearance of a variable* x *in a* Σ*-term* t *is* guarded *if there is a subterm* $c \cdot s$ *in* t *with* c *being an atomic action, and* s *being a term that contains the mentioned appearance of* x.

Definition 13. *Let* $\Sigma \supseteq \Sigma_{BPA}$ *be a signature and* $E \supseteq E_{BPA}$ *a set of axioms. A term* t, *defined over* (Σ, E), *is said to be* completely guarded *if all the appearances of all the variables in it are guarded. A term* t *is said to be* guarded *if only completely guarded terms can be derived from* t *by using the axioms from* E.

Definition 14. *Let* (Σ, E) *be a specification such that* $\Sigma \supseteq \Sigma_{BPA}$ *and* $E \supseteq E_{BPA}$. *We say that a recursive specification:*

$$x = s_x(V), \qquad \textit{where } x \in V$$

defined over (Σ, E) *is* completely guarded *if every* s_x *is* guarded, *and it is guarded if it is equivalent to some completely guarded recursive specification over* (Σ, E).

Definition 15. *Let* $\mathbf{A}$ *be a* Σ*–algebra, for some* $\Sigma \supseteq \Sigma_{BPA}$. *The element* $p \in A$ *is said to be* definable *if it is solution of some guarded specification.*

Definition 16 (PA_ε**–algebra**). *Consider a signature* Σ_{PA} *with set of atomic actions* $C \supseteq \{\delta, \varepsilon)\}$, *unary operator* $\surd$, *binary operators* $\lfloor\!\lfloor$, $\|$, $\cdot$ *and* $+$ *(operators are listed by the order of their of priority). Then, any* Σ_{PA}*–algebra which is model of the following system of axioms, is said to be a* PA_ε*–algebra:*

$$\begin{aligned}
x + y &= y + x \\
(x + y) + z &= x + (y + z) \\
x + x &= x \\
(x + y)z &= xz + yz \\
x(yz) &= (xy)z
\end{aligned}$$

$$\begin{aligned}
\delta + x &= x \\
\delta x &= \delta \\
x\varepsilon &= x \\
\varepsilon x &= x
\end{aligned}$$

$$\begin{aligned}
x \| y &= x \lfloor\!\lfloor y + y \lfloor\!\lfloor x + \surd(x) \cdot \surd(y) \\
\varepsilon \lfloor\!\lfloor x &= \delta \\
cx \lfloor\!\lfloor y &= c(x \| y) \qquad \forall c \in C \setminus \{\varepsilon\} \\
(x + y) \lfloor\!\lfloor z &= x \lfloor\!\lfloor z + y \lfloor\!\lfloor z
\end{aligned}$$

$$\begin{aligned}
\surd(\varepsilon) &= \varepsilon \\
\surd(c) &= \delta\,, \qquad \forall c \in C \setminus \{\varepsilon\} \\
\surd(x + y) &= \surd(x) + \surd(y) \\
\surd(xy) &= \surd(x)\,\surd(y)
\end{aligned}$$

Clearly, $\Sigma_{BPA} \subseteq \Sigma_{PA}$, and it is obvious that every PA_ε algebra is model for the axions of E_{BPA}. Therefore, it can be assumed that any PA_ε algebra is also a BPA algebra.

3. Some properties of PA_ε algebras

Let $\mathbf{A}$ be a PA_ε algebra with carrier A. In general, two or more atomic actions might be interpreted in $\mathbf{A}$ with a same element of A. These interpretations are not of our intereset because then $\mathbf{A}$ can be considered as an example of PA_ε algebra with fewer atomic actions. Especially, if some atomic action c has the same interpretation as ε, i.e. $\overline{c} = \overline{\varepsilon}$, then for every

$p \in A$ it follows that $p = \overline{\varepsilon}\, p = \surd(\overline{\varepsilon})p = \surd(\overline{c})p = \overline{\delta}\, p = \overline{\delta}$. Consequently, $\overline{c} = \overline{\varepsilon}\overline{\delta}$ and $A = \{\overline{\delta}\}$.

Further on, only algebras with different interpretation of atomic actions will be considered. Also, the symbols of the atomic actions will be used to represent their interpretations. It will be assumed (unless otherwise stated) that the set A is the carrier for a PA_ε algebra $\mathbf{A}$.

Proposition 1. *The groupoid* $(A, +)$ *is semi-lattice, with the relation* $\leq$ *defined by:*

$$p \leq q \Leftrightarrow p + q = q$$

being on ordering on A. *In this ordering, for every* $p, q \in A$, $sup\{p, q\}$ *exists and is equal to* $p + q$. □

With respect to the ordering defined in the proposition 1, every finite PA_ε algebra has the largest element, and it is simply the sum of all the elements of A.

Proposition 2. *If* $\mathbf{A}$ *is finite* PA_ε *algebra, then* $(A, \leq)$ *is lattice.*

Proof. Let M be a nonempty subset of A, and M_* be the set of minorants for M in A. M_* is finite and nonempty ($\delta \in M_*$). Then $m = \sum_{q \in M_*} q$ is in M_* because, for every $p \in M$, the following equation holds:

$$p + m = p + \sum_{q \in M_*} q = \sum_{q \in M_*} (p + q) = \sum_{q \in M_*} p = p\,.$$

□

It should be noticed that the proposition 2 may not hold in the case of infinite PA_ε algebras.

Proposition 3. *If* p *and* q *are elements of* A *and* $\varepsilon \leq p$, *then* $q \leq p \cdot q$. *Especially, if* q *is the largest element in the algebra, then* $p \cdot q = q$.

Proof. If $\varepsilon \leq p$ then $p + \varepsilon = p$. Thus, $p \cdot q + q = p \cdot q + \varepsilon \cdot q = (p + \varepsilon) \cdot q = p \cdot q$, i.e. $q \leq p \cdot q$. □

Example 1. If q is the largerst element in the nontrivial PA_ε algebra A, but $\varepsilon \not\leq p$, the equation $p \cdot q = q$ does not have to hold. For example, if $p = \delta$ then $\varepsilon \not\leq \delta$ and $\delta \cdot q = \delta \neq q$.

Proposition 4. *The atomic action* δ *is the smallest element of* A. □

Proposition 5. For every finite element $p \in A$, the following equations hold:

$$\text{(i)}\ \ \surd(p) = \begin{cases} \varepsilon, & if \quad \varepsilon \leq p \\ \delta, & otherwise \end{cases} \qquad \text{(ii)}\ \ p = p \lfloor\!\lfloor \varepsilon + \surd(p)$$ □

The proof of the proposition 5 requires more elaborate considerations that are not going to be given in this paper. These, as well as several other properties considering the operators of the PA_ε algebras, can be found in [1], pages 76 – 77.

Proposition 6. *If $p \in A \setminus \{\delta, \varepsilon\}$ and $p \leq \varepsilon$, then p is infinite element.*

Proof. Let p be a finite element of $A \setminus \{\delta, \varepsilon\}$ and $p \leq \varepsilon$. Then $p \neq \varepsilon$ and, according to the proposition 5(i), there must be $\surd(p) = \delta$. But then, from 5(ii), it can be inferred that $p = p \| \varepsilon + \surd(p) = p \| \varepsilon + \delta = p \| \varepsilon + \varepsilon \| \varepsilon = (p+\varepsilon) \| \varepsilon = \varepsilon \| \varepsilon = \delta$. □

Proposition 7. *For every element p of the non-trivial PA_ε algebra A, if $\varepsilon \leq p$, then $\surd(p) \neq \delta$.*

Proof. $\varepsilon \leq p \Rightarrow p + \varepsilon = p \Rightarrow \surd(p) + \surd(\varepsilon) = \surd(p) \Rightarrow \surd(p) + \varepsilon = \surd(p) \Rightarrow \surd(p) \neq \delta$. □

Proposition 8. *If $p \in A \setminus \{\delta, \varepsilon\}$ is interpretation of atomic action, then p is not comparable to ε.*

Proof. From the conditions of the proposition, p is finite element and $\surd(p) = \delta$. Now, the conclusion follows from the propositions 6 and 7. □

Proposition 9. *The set of finite elements of* **A** *forms a subalgebra of* **A**. □

Proposition 10. *There is no nontrivial PA_ε algebra, other then $\{\delta, \varepsilon\}$, whose every element is interpretation of atomic action.*

Proof. Let **A** be a nontrivial PA_ε algebra, different from $\{\delta, \varepsilon\}$, with all of its elements being interpretations of atomic actions. If $p \in A \setminus \{\delta, \varepsilon\}$, then p and $q = p + \varepsilon$ are both images of atomic actions and, moreover, $q + \varepsilon = q$ i.e. $\varepsilon \leq q$. Thus, q is comparable to ε, so according to proposition 8, it must be $q = \delta$ or $q = \varepsilon$. In the first case, $\varepsilon \leq \delta$ and consequently $\varepsilon = \delta$, a contradiction to the assumed non-triviality of **A**. In the second case, $p + \varepsilon = \varepsilon$, and this is incompatible with the proposition 8. □

Proposition 11. *If $p \neq q \in A$ are finite and $\surd(p) = \surd(q)$, then $p + \varepsilon \neq q + \varepsilon$.*

Proof. It follows from $p + \varepsilon = q + \varepsilon$ and the proposition 5(ii), that $p = p \| \varepsilon + \surd(p) = (p+\varepsilon) \| \varepsilon + \surd(p) = (q+\varepsilon) \| \varepsilon + \surd(q) = q \| \varepsilon + \surd(q) = q$. □

Proposition 12. *Suppose that $\mathbf{A}_1$ and $\mathbf{A}_2$ are two isomorphic PA_ε algebras and $\zeta \colon \mathbf{A}_1 \to \mathbf{A}_2$ is isomorphism. If ε_1 and ε_2 are interpretations of the atomic action ε in both of the algebras, then $\zeta(\varepsilon_1) = \varepsilon_2$.*

Proof. Let $\varepsilon_2 = \zeta(p)$ for some $p \in A_1$. Then $\zeta(\varepsilon_1) = \zeta(\varepsilon_1) \cdot \varepsilon_2 = \zeta(\varepsilon_1) \cdot \zeta(p) = \zeta(\varepsilon_1 \cdot p) = \zeta(p) = \varepsilon_2$. □

Proposition 13. *Let ζ be an automorphism on a finite PA_ε algebra* **A**. *If $p \neq q$ are comparable elements of A, then $\zeta(p) \neq q$.*

Proof. Let $p \neq q \in A$ and $\zeta(p) = q$. Since ζ does not affect the ordering of A, the sequence $p, \zeta(p), \zeta^2(p), \ldots, \zeta^n(p), \ldots$ is strictly ascending or descending, depending whether $p < q$ or $q < p$. Anyway, A would have to have infinitely many elements. □

4. Finite PA_ε algebras and their generation

According to the PA_ε axioms, the interpretation of every term consisted only of atomic actions δ and ε, must be either δ or ε. Therefore, a PA_ε algebra that has only these two constants in its signature, can not have other finite elements. As a consequence, the following propositions can be derived.

Proposition 14. *There is no three-element PA_ε algebra whose all the elements are finite.*

Proof. The finiteness of the third element (the one different from δ and ε) implies that it must be an atomic action. But that is opposed to the proposition 10. □

Proposition 15. *For each integer $n \geq 3$, there is no n-element PA_ε algebra with $n-3$ infinite elements.*

Proof. If there is an n-element PA_ε algebra with $n-3$ infinite elements, there would exist a three-element PA_ε algebra whose all the elements are finite. But, this is inconsistent with the proposition 14. □

4.1. Three-elements algebras

Here, PA_ε algebras with only three elements will be considered.

Proposition 16. *Every three-element PA_ε algebra is isomorphic to some algebra defined over some of the lattices from figure* 1. *The algebras over these lattices are pairwise nonisomorphic.*

Proof. For every non-trivial PA_ε algebra, there must be $\delta < \varepsilon$. The only three-element lattices that satisfy this condition are depicted on figure 1. It follows directly from the propositions 12 and 13 that they are not isomorphic. □

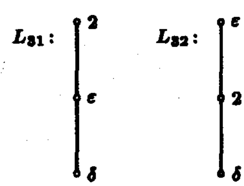

Fig. 1. Two lattices over $A = \{\delta, \varepsilon, 2\}$

The database of all the PA_ε algebras that can be constructed over the lattices from figure 1, is generated with a computer program that utilizes already devised properties. The results are summarized in the following assertion.

Proposition 17. *There are* 86 *nonisomorphic three-element PA_ε algebras with one infinite element. Out of that,* 81 *are generated over the lattice L_{31}, and* 5 *are generated over the lattice L_{32}* □

Example 2.

$\cdot$	δ	ε	2
δ	δ	δ	δ
ε	δ	ε	2
2	δ	2	2

The above table presents the diagram of the operation $\cdot$ of one three-element PA_ε algebra over the lattice L_{32}. Its other operations are defined as: $p \lfloor\!\lfloor q = \delta$, $\surd(p) = p$, $p \| q = p \lfloor\!\lfloor q + q \lfloor\!\lfloor p + \surd(p)\,\surd(q)$, for every $p, q \in A$. The element 2 is infinite. It should be noticed that in this example $p \cdot q = \inf\{p, q\}$. □

4.2. Four-element algebras

As stated by proposition 15, a four-element PA_ε algebra may have two infinite elements, or may have no infinite elements at all. In the later case, according to proposition 10, three of the elements must be atomic actions, while the fourth is interpretation of a finite term.

Figure 2 depicts several four-element lattices that may represent the + operation of a PA_ε algebra. Simple examination of all the four-element lattices, proves the following statement.

Proposition 18. *Every for element PA_ε algebra is isomorphic to same algebra defined over some of the lattices from figure 2.* □

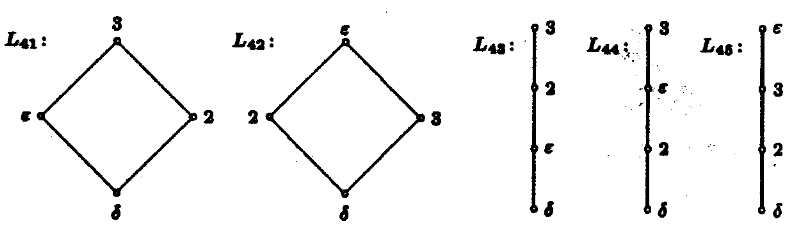

Fig. 2. Lattices over $A = \{\delta, \varepsilon, 2, 3\}$

Proposition 19. *The lattice L_{41} is (up to isomorhism) the only four-element lattice over which a PA_ε algebra with no infinite elements might be generated.*

Proof. In all the lattices from the figure 2, except the lattice L_{41}, the element ε is comparable to every other element. Thence, by the proposition 8, no element in these lattices, other then δ and ε, can be interpretations of atomic actions. But then, the only finite elements that the algebra may have, are δ and ε. □

From the proposition 12, it is obvious that the algebras generated over lattices L_{41} and L_{42}, can not be isomorphic. Namely, ε is the largest element in the second lattice, but not in the first. The same may be concluded for the algebras over L_{43}, L_{44} and L_{45}. Besides, according to proposition 13, no two algebras over the lattice L_{41} are isomorphic since the only two incomparable elements in that lattice are ε and 2, and by proposition 12, they can not be mapped one into another. The same holds for the algebras over the lattices L_{43}, as well.

Hawever, these propositions are not applicable to the lattice L_{42}. For this lattice, there is an automorphism that interchanges the incomparable elements 2 and 3. Therefore, after the generation of PA_ε algebras over this lattice, it is necessary to additionally discard the isomorphic ones.

The results of the computer-based generation of the four-element PA_ε algebras are presented in the following propositions.

Proposition 20. *There exist* 201829 *nonisomorphic four-element PA_ε algebras with two infinite elements. Out of that,* 5376 *are generated over the lattice L_{41},* 8 *over the lattice L_{42},* 190000 *over the lattice L_{43},* 6400 *over L_{44}, and* 45 *over the lattice L_{45}.*

Examle 3. There is an algebra over the lattice L_{42}, constructed in similar manner as the algebra in example 2. Its operations are defined as $p \cdot q = \inf\{p, q\}$, $p \lfloor\!\lfloor q = \delta$, $\surd(p) = p$, $p \| q = p \lfloor\!\lfloor q + q \lfloor\!\lfloor p + \surd(p)\,\surd(q)$, for every $p, q \in A$. The elements 2 and 3 are infinite.

It can be easily shown that a PA_ε algebras of any finite cardinality can be constructed in this way. It suffices to choose a lattice in which δ is the smallest, ε is the largest, and all the other elements are pairwise incomparable and infinite. Other operations are defined as in this example.

□

Proposition 21. *There are* 3 *nonisomorphic four-element PA_ε algebras with no infinite elements. All of them are generated over the lattice L_{41}, with atomic actions δ, ε and* 2. *The diagrams of their operations are:*

1.

$\cdot$	δ	ε	2	3
δ	δ	δ	δ	δ
ε	δ	ε	2	3
2	δ	2	δ	2
3	δ	3	2	3

$\lfloor\!\lfloor$	δ	ε	2	3
δ	δ	δ	δ	δ
ε	δ	δ	δ	δ
2	δ	2	δ	2
3	δ	2	δ	2

	$\surd$
δ	δ
ε	ε
2	δ
3	ε

2.

$\cdot$	δ	ε	2	3
δ	δ	δ	δ	δ
ε	δ	ε	2	3
2	δ	2	2	2
3	δ	3	2	3

$\lfloor\!\lfloor$	δ	ε	2	3
δ	δ	δ	δ	δ
ε	δ	δ	δ	δ
2	δ	2	2	2
3	δ	2	2	2

	$\surd$
δ	δ
ε	ε
2	δ
3	ε

3.

$\cdot$	δ	ε	2	3
δ	δ	δ	δ	δ
ε	δ	ε	2	3
2	2	2	2	2
3	2	3	2	3

$\Vert$	δ	ε	2	3
δ	δ	δ	δ	δ
ε	δ	δ	δ	δ
2	2	2	2	2
3	2	2	2	2

	$\surd$
δ	δ
ε	ε
2	δ
3	ε

The operation $\Vert$ *can be derived from the other operations of the algebra. Note the noncommutativity of the operation* $\cdot$ *in* 3. □

4.3. Five-element algebras

Using the previous propositions, it is easy to determine all nonisomorphic five-element lattice that are appropriate for generation of PA_ε algebras. In this paper, only five-element algebras with at least one atomic action, aside from the necessary δ and ε, are considered.

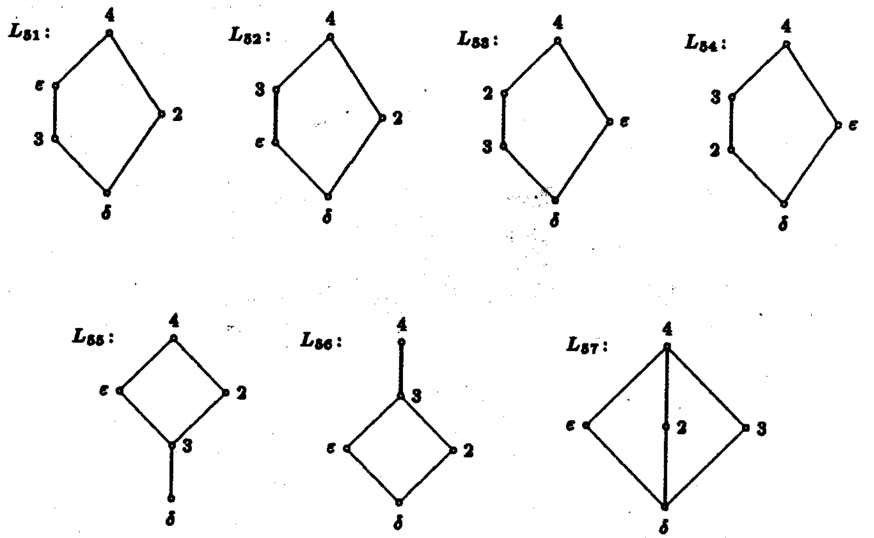

Fig. 3. Lattices over $A = \{\delta, \varepsilon, 2, 3, 4\}$, suitable to PA_ε algebras with atomic action 2.

Proposition 22. *Every five-element* PA_ε *algebra with at least one atomic action different from* δ *and* ε, *is isomorphic to some algebra defined over some of the lattices from figure* 3. (*In the depicted lattices, that atomic action is the element* 2.) *The algebras are pairwise nonisomorphic.*

Proof. As stated in proposition 4, δ must be the smallst element of the lattice. From proposition 8, ε and the atomic action 2 are incomparable. Hence, none of them can be the largest element of the lattice. To avoid the isomorphness, the element 4 may be chosen to be the largest element. Only the lattices from figure 3 comply with this framework.

It remains to be shown that among the obtained algebras, there won't be isomorphic ones. From the propositions 12 and 13, it follows directly that between any two algebras over the latices L_{51}, L_{52}, L_{53}, L_{54}, L_{55} and L_{56} there is no isomorphism. Neither could an algebra over the lattice L_{57} be isomorphic to an algebra over some other lattice. Besides, according to the proposition 13, the only mapping that could be automorphism on the lattice L_{57}, is the permutation (2 3). But,

$$\surd(3) = \surd(3) + \delta = \surd(3) + \surd(2) = \surd(3+2) = \surd(4) = \surd(\varepsilon+2) = \varepsilon+\delta = \varepsilon \neq \delta = \surd(2),$$

so this mapping is not sound with the operation $\surd$. Therefore, every two algebras over the lattice L_{57} are also nonisomorphic. □

Proposition 23. *There is no five-element PA_ε algebra with two atomic actions different from δ and ε.*

Proof. If there is a five-element PA_ε algebra with two atomic actions besides δ and ε, then it must be constructed over a lattice that is isomorphic with some of the lattices from figure 3. Without loss of generality, it can be assumed that this algebra is constructed precisely over some lattice from the figure 3, and that the element 2 is one of its atomic actions. According to what is stated in proposition 8, the element 4 can not be another atomic action, because it is comparable with ε. The same argument holds for the element 3 in the lattice L_{51}, L_{52} and L_{55}. In the other lattices, 3 is not comparable with ε, but $3 + \varepsilon = 2 + \varepsilon$. If the assumption that 3 is the second atomic action is correct, then the equation $\surd(3) = \delta = \surd(2)$ must also hold. However, this is in contradiction with the proposition 11. □

The results of the computer-based generation of the five-element PA_ε algebras, are presented in the following proposition.

Proposition 24. *There are* 2607 *nonisomorphic five-element PA_ε algebras with three atomic actions (δ, ε and* 2*). Out of that,* 6 *are generated over the lattice L_{51},* 144 *are generated over L_{52},* 13 *over L_{53},* 20 *over L_{54},* 16 *over L_{55},* 2403 *over L_{56}, and* 5 *over L_{57}.* □

Example 4 Figure 4 depicts one of the five-element algebras over the lattice L_{57}. The algebra is represented with the diagrams of its operations. It has three atomic actions: δ, ε and 2. The element 4 is finite, since $4 = 2 + \varepsilon$, while the element 3 is infinite. □

$\cdot$	δ	ε	2	3	4
δ	δ	δ	δ	δ	δ
ε	δ	ε	2	3	4
2	2	2	2	2	2
3	2	3	2	4	4
4	2	4	2	4	4

$\Vert\!\!\lfloor$	δ	ε	2	3	4
δ	δ	δ	δ	δ	δ
ε	δ	δ	δ	δ	δ
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2

	$\surd$
δ	δ
ε	ε
2	δ
3	ε
4	ε

Fig. 4. Diagrams of the operations of PA_ε algebra over the lattice L_{57}, with atomic action 2

The obtained database of five-element algebras can help answering some questions about the characteristics of PA_ε algebras. So, for instance, the following proposition holds:

Proposition 25. *There exists a five-element PA_ε algebra with an infinite, definable element.* □

The proof of the proposition 25 will result from the subsequent considerations. From the definition 15, it is clear that infinite, but definable elements may exist only in algebras with other atomic actions, besides δ and ε. Therefore, the example that proves the proposition 25, must be sought for among five-element algebras having three atomic actions (δ, ε and 2). To check over the definability of the elements of these algebras, the following proposition may be used.

Proposition 26. *In a five-element PA_ε algebra* **A** *with atomic action 2, an infinite element is definable if and only if it is a solution of the recursive specification $x = 2 \cdot x$.*

Proof. If $p \in A$ satisfies the equation $p = 2 \cdot p$ then by definition, p is definable. Contrary, let p be infinite and definable element. In a five-element PA_ε algebra, there can not be two infinite elements (according to the proposition 15), so p must be the only one. Furthermore, p is definable, so it must be a solution of some guarded recursive specification. Every quarded recursive specification in algebra with atomic actions δ, ε and 2 is equivalent to some of the specifications:

$$\begin{cases} x_1 = \Sigma 2 \cdot x_i \\ x_i = t_i, \qquad x_i \in V \end{cases} \quad \text{or} \quad \begin{cases} x_1 = \Sigma 2 \cdot x_i + \varepsilon \\ x_i = t_i, \qquad x_i \in V \end{cases}$$

where t_i are guarded terms, and x_1 is the root variable (vatiable that corresponds to p). Now, all the finite elements from the sums $\Sigma 2 \cdot x_i$ and $\Sigma 2 \cdot x_i + \varepsilon$ may be discarded. Namely, the remaining of the sums will still be infinite element, and p is the only alike. Therefore, the above specifications have the same solution as:

$$\begin{cases} x_1 \Sigma 2 \cdot x_i \\ x_i = t_i, \qquad x_i \in V' \end{cases}$$

where V' contains only those variables from V that, in the complete solutions of the above specifications, corresponds to the infinite elements. Again, p is the only infinite element, so the previous specification has the same solution as the specification:

$$x = \Sigma 2 \cdot x \, .$$

But, + is idempotent, so this is equivalent to:

$$x = 2 \cdot x \, .$$

Therefore, $p = 2 \cdot p$. □

$\cdot$	δ	ε	2	3	4
δ	δ	δ	δ	δ	δ
ε	δ	ε	2	3	4
2	δ	2	2	3	2
3	δ	3	δ	δ	δ
4	δ	4	2	3	4

$\lfloor\!\lfloor$	δ	ε	2	3	4
δ	δ	δ	δ	δ	δ
ε	δ	δ	δ	δ	δ
2	δ	2	2	2	2
3	δ	2	2	2	2
4	δ	2	2	2	2

	$\surd$
δ	δ
ε	ε
2	δ
3	δ
4	ε

Fig. 5.
Diagrams of the operations of PA_ε algebra over the lattice L_{53}, with atomic action 2 and infinite, definable element 3

Example 5. Diagrams from figure 5 depict the operations $\cdot$, $\lfloor\!\lfloor$ and $\surd$ of an algebra with carrier $A = \{\delta, \varepsilon, 2, 3, 4\}$, constructed over the lattice L_{53}. This algebra has an infinite and definable element 3. The diagram of $\|$ is derived from the other operations. □

Computer-based examination of all five-element PA_ε algebras with atomic action 2, proves the validity of the following statement.

Proposition 27. *There are 14 five-element PA_ε algebras with infinite, definable element. Out of that,* 6 *are generated over the lattice L_{53},* 2 *are generated over L_{54} and* 6 *are generated over the lattice L_{55}.*

All of them have δ, ε and 2 *as atomic actions, while the infinite and definable element is* 3. □

5. Conclusion

This paper presents some basic characteristics of PA_ε algebras. These characteristics are then used in derivation of several propositions of three, four and five-element PA_ε algebras that, applied in a computer program, greatly simplified the generation of databases of these algebras.

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КОНЕЧНИ ПРОЦЕСНИ АЛГЕБРИ

Александар Наневски

Резиме

Процесните алгебри PA_ε се структури со посебна важност при теоретските разгледувања на паралелните процеси, како математичка основа за дефинирање на паралелните процеси. Со цел за доближување до реалните процеси, овие алгебри вообичаено се збогатуваат со повеќе операции. Така, покрај основните операции за одлучување $(+)$ и проследување $(\cdot)$ се додаваат операции за лева паралелност, паралелност, сигурно темпирање, ќорсокак и др. Сето ова го зголемува и бројот на аксиомите, така што се поставува проблем за наоѓање на погодни конечни модели за нив. Во трудот, за процесни алгебри со 3, 4 и 5 елементи, е даден нивниот комплетен опис.

Faculty of Natural Sciences and Mathematics,

Institute of Informatics,

St. Cyril and Methodius University,

P. Box 162
91 000 Skopje

Macedonia