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CONTRIBUTIONS TO MODELING OF 4D FRACTALS

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Abstract. Affine invariant IFS's (AIFS) are Iterated Function Systems whose affine transformations are based on areal coordinates with respect to a certain simplex. In this note, we consider 4-dimensional generalizations of some classic and non-classic fractal objects generated by such AIFS and show how it can be used for modeling purposes. The stress is put on the one of the most desirable properties in modeling, according to CAGD specialists, convex hull property.

1. Introduction

The Affine invariant Iterated Function System (AIFS) introduced and studied by Kocić and Simoncelli ([4], [5], [6]) has been shown to be a very promising tool for fractal sets modeling. At first, a brief introduction about the notion of IFS, in general, and especially of AIFS, will be given.

Let $\{w_i \,|\, i = 1, 2, \ldots, n\}$ be a set of contractive affine mappings defined on the complete Euclidian metric space $(\mathbb{R}^m, d_E)$ $(m > 1)$

$$w_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \quad \mathbf{x} \in \mathbb{R}^m, \quad i = 1, 2, \ldots, n,$$

where $\mathbf{A}_i$ are $m \times m$ real matrices and $\mathbf{b}_i$ are m-dimensional real vectors. The system $S = \{\mathbb{R}^m; w_1, w_2, \ldots, w_n\}$, $\quad n \geq 2$ is called (*hyperbolic*) *Iterated Function System* (*IFS*). The *Hutchinson operator*, $W_S : \mathcal{H}(\mathbb{R}^m) \to \mathcal{H}(\mathbb{R}^m)$, defined by

$$W_S(B) = \bigcup_{i=1}^{n} w_i(B),$$

and associated to the IFS, is shown to be a contractive mapping on the complete metric space $(\mathcal{H}(\mathbb{R}^m), h)$, where $\mathcal{H}(\mathbb{R}^m)$ denotes the space of nonempty compact subsets of $\mathbb{R}^m$ and h is Hausdorff metric induced by d_E [2]. Accordingly, W_S has a unique fixed point, $A = W_S(A) \in \mathcal{H}(\mathbb{R}^m)$, called the *attractor* of the IFS S.

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Definition 1.1. A (non-degenerate) *m-dimensional simplex* $\hat{\mathbf{P}}_m$ is the convex hull of a set of m+1 affinely independent points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{m+1}$ in the Euclidean space of dimension m or higher, $\hat{\mathbf{P}}_m = \mathrm{conv}\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{m+1}\}$. The vertices of $\hat{\mathbf{P}}_m$ will be denoted by $\mathbf{P}_m$ and represented by the matrix $\mathbf{P}_m = \left[\mathbf{p}_1^T \, \mathbf{p}_2^T \, \dots, \mathbf{p}_{m+1}^T\right]^T$. If the vectors $\{\mathbf{p}_i\}$ coincide with the orthonormal basis in $\mathbb{R}^m$, $\mathbf{e}_i = [\delta_{ij}]_{j=1}^m$, $i = 1, 2, \dots, m$, the simplex is called *standard*.

Let $S = [s_{ij}]_{i,j=1}^{m+1}$ be an $(m+1) \times (m+1)$ row-stochastic real matrix (its rows sum up to 1).

Definition 1.2. We refer to the linear mapping $\mathcal{L} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, such that $\mathcal{L}(\mathbf{x}) = \mathbf{S}^T \mathbf{x}$ as the *linear mapping associated with* $\mathbf{S}$.

Definition 1.3. Let $\hat{\mathbf{P}}_m$ be a non-degenerate simplex and let $\{\mathbf{S}_i\}_{i=1}^n$ be a set of real square nonsingular row-stochastic matrices of order $m+1$. The system $\Omega(\hat{\mathbf{P}}_m) = \{\hat{\mathbf{P}}_m; \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_n\}$ is called (*hyperbolic*) *Affine invariant IFS* (*AIFS*), provided that the linear mappings associated with $\mathbf{S}_i$ are contractions in $(\mathbb{R}^m, d_E)$ ([4], [5], [6]).

According to CAGD (Computer Aided Geometric Design) literature, any CAGD mathematical model should have several properties that make it a real free-form model. The AIFS fractal model possesses several of them: affine invariance property (one of the essential properties), convex hull property, continuity property, interpolation property, symmetry property and finally, numerically stable and simple algorithm for generating the fractal set. In this paper we will keep our attention to the convex hull property, later practicing the theory in a very special case, on the examples of four-dimensional fractals.

2. Switching from AIFS to IFS and vice versa

In the definition of the AIFS $\Omega(\hat{\mathbf{P}}_m) = \{\hat{\mathbf{P}}_m; \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_n\}$ the simplex $\hat{\mathbf{P}}_m$ considers to be a standard one (since any simplex generates an affine space, in which, the areal (barycentric) coordinates of the vertices of that simplex are $(m+1)$-tuples: $(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$). The standard simplex will be denoted by $\hat{\mathbf{T}}_m = \mathrm{conv}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{m+1}\}$. The system $\Omega(\hat{\mathbf{T}}_m)$ acts on the affine space $\mathbb{V}^m = \mathrm{conv}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{m+1}\} \subset \mathbb{R}^{m+1}$ and all transformations defined by this system, $\mathcal{L}_i : \mathbf{x} \mapsto \mathbf{S}_i^T \mathbf{x}$, $\mathbf{x} \in \mathbb{V}^m$, $i = 1, 2, \dots, n$ are linear. Finding affine transformation $\mathcal{A}_i : \mathbf{x} \mapsto \mathbf{A}_i \mathbf{x} + \mathbf{b}_i$, $\mathbf{x} \in \mathbb{V}^m$, $i = 1, 2, \dots, n$, that performs the same mapping as $\mathcal{L}_i$, will enable us to bring into play the deeply developed theory of IFS and therefore using all advantages of both IFS and AIFS in fractal modeling.

In [7], it was shown that

$$\left[\begin{array}{c} \mathbf{A}^T \\ \hline \mathbf{b}^T \end{array}\right] = \left[\overline{\mathbf{E}}_{m+1} \left(\mathbf{Q}_m^{-1}\right)^T \,|\, \mathbf{1}\right]^{-1} \overline{\mathbf{S}} \left(\mathbf{Q}_m^{-1}\right)^T \qquad \text{and} \tag{1}$$

$$\overline{\mathbf{S}} = \left[\overline{\mathbf{E}}_{m+1} \left(\mathbf{Q}_m^{-1}\right)^T \mid \mathbf{1}\right] \left[\begin{array}{c} \mathbf{A}^T \\ \hline \mathbf{b}^T \end{array}\right] \mathbf{Q}_m^T \tag{2}$$

take place, where

$$\bar{\mathbf{E}}_{m+1} = \left[\begin{array}{c} \mathbf{E}_m \\ \hline \mathbf{0}^T \end{array}\right] = \begin{bmatrix} 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 1 & & 0 & 1 \\ 0 & 0 & & 0 & 0 \end{bmatrix},$$

$$\mathbf{Q}_m = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2\cdot 3}} & -\frac{1}{\sqrt{3\cdot 4}} & \cdots & -\frac{1}{\sqrt{m(m+1)}} \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3\cdot 4}} & \cdots & -\frac{1}{\sqrt{m(m+1)}} \\ 0 & 0 & \sqrt{\frac{3}{4}} & \cdots & -\frac{1}{\sqrt{m(m+1)}} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \sqrt{\frac{m}{m+1}} \end{bmatrix},$$

$$\mathbf{Q}_m^{-1} = \begin{bmatrix} \sqrt{\frac{2}{1}} & \frac{1}{\sqrt{1\cdot 2}} & \frac{1}{\sqrt{1\cdot 2}} & \cdots & \frac{1}{\sqrt{1\cdot 2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2\cdot 3}} & \cdots & \frac{1}{\sqrt{2\cdot 3}} \\ 0 & 0 & \sqrt{\frac{4}{3}} & \cdots & \frac{1}{\sqrt{3\cdot 4}} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \sqrt{\frac{m+1}{m}} \end{bmatrix}.$$

and $\bar{\mathbf{S}}$ is $\mathbf{S}$ without its last column. The $m \times m$ square matrix $\mathbf{A}$ and the m-dimensional vector $\mathbf{b}$ define affine transformation $\mathcal{A}$ and the row stochastic $(m+1) \times (m+1)$ matrix $\mathbf{S}$ defines linear transformation $\mathcal{L}$, that is equivalent transformation to $\mathcal{A}$.

Two theorems that follow give explicit formulas that connect the items of the affine transformation $\mathcal{A}$ of the hyper-plane $\mathbb{V}^m$ with the items of the linear transformation $\mathcal{L}$ of the same hyper-plane.

Theorem 2.1. *The explicit formula for the items of the matrix* $\mathbf{A}$ *and vector* $\mathbf{b}$ *in terms of the items of the matrix* $\mathbf{S}$*, reads*

(3)
$$\begin{cases} a_{ij} &= -\dfrac{1}{\sqrt{i(i+1)}\sqrt{j(j+1)}}\left[i\sum_{p=1}^{j-1} s_{pi} - jis_{ji} + \right. \\ &+ \left. is_{m+1,i} + \sum_{q=i}^{m}\sum_{p=1}^{j-1} s_{pq} + \sum_{q=i}^{m} s_{m+1,q} - j\sum_{q=i}^{m} s_{jq}\right]; \\ b_j &= \dfrac{1}{\sqrt{j(j+1)}}\left[js_{m+1,j} + \sum_{q=j}^{m} s_{m+1,q}\right], \; i,j = 1,2,\ldots,m. \end{cases}$$

Proof. The formula (1) can be rewritten as $\begin{bmatrix} \mathbf{A}^T \\ \hline \mathbf{b}^T \end{bmatrix} = (\mathbf{T}^*_m)^{-1} \cdot \mathbf{T}'_m$, where $\mathbf{T}^*_m = \left[\overline{\mathbf{E}}_{m+1}\left(\mathbf{Q}_m^{-1}\right)^T \mid \mathbf{1}\right]$ and $\mathbf{T}'_m = \overline{\mathbf{S}}\left(\mathbf{Q}_m^{-1}\right)^T$. Let $(\mathbf{T}^*_m)^{-1} = [\boldsymbol{\tau}_1\, \boldsymbol{\tau}_2\, \ldots\, \boldsymbol{\tau}_{m+1}]$ and $\mathbf{T}'_m = [\boldsymbol{\tau}'_1\, \boldsymbol{\tau}'_2\, \ldots\, \boldsymbol{\tau}'_{m+1}]$. Then, $a_{ij} = (\boldsymbol{\tau}_i)^T \cdot \boldsymbol{\tau}'_j$ and $b_j = (\boldsymbol{\tau}_{m+1})^T \cdot \boldsymbol{\tau}'_j$, $i,j = 1,2,\ldots,m$. It checks out without difficulties that

$$\begin{cases} \boldsymbol{\tau}_i^T &= \frac{1}{\sqrt{i(i+1)}}\left[-1,\ldots,-1,\underbrace{i}_{i-\text{th}},0,\ldots,0,-1\right], \; i = 1,2,\ldots,m; \\ \boldsymbol{\tau}_{m+1}^T &= [0,0,\ldots,0,1]; \end{cases}$$

$$(\boldsymbol{\tau}'_j)^T = \frac{1}{\sqrt{j(j+1)}}\left[js_{1j} + \sum_{q=j+1}^{m} s_{1q},\; js_{2j} + \sum_{q=j+1}^{m} s_{2q},\; \ldots, js_{m+1,j} + \sum_{q=j+1}^{m} s_{m+1,q}\right].$$

Taking product yields (3). □

Theorem 2.2. *The explicit formula for the items of the matrix* $\mathbf{S}$ *in terms of the items of the matrices* $\mathbf{A}$ *and vector* $\mathbf{b}$ *is given by*
(4)
$$s_{ij} = \begin{cases} \sqrt{\dfrac{j}{j+1}}\left[\sum_{q=1}^{i-1}\dfrac{a_{jq}}{\sqrt{q(q+1)}} + \sqrt{\dfrac{i+1}{i}}a_{ji} + b_j\right] - \\ - \sum_{p=j+1}^{m}\dfrac{1}{\sqrt{p(p+1)}}\left[\sum_{q=1}^{i-1}\dfrac{a_{pq}}{\sqrt{q(q+1)}} + \sqrt{\dfrac{i+1}{i}}a_{pi} + b_p\right], \; i \le m; \\ \sqrt{\dfrac{j}{j+1}}b_j - \sum_{p=j+1}^{m}\dfrac{b_p}{\sqrt{p(p+1)}}, \; i = m+1; \end{cases}$$

for $j = 1,2,\ldots,m$.

Proof. This time the formula (2) is used, $\mathbf{S} = \mathbf{T}^*_m \cdot \mathbf{Q}'_m$, where $\mathbf{T}^*_m = \left[\overline{\mathbf{E}}_{m+1}\left(\mathbf{Q}_m^{-1}\right)^T \mid \mathbf{1}\right]$ and $\mathbf{Q}'_m = \begin{bmatrix} \mathbf{A}^T \\ \hline \mathbf{b}^T \end{bmatrix} \mathbf{Q}_m^T$. Denoting the i-th row of the matrix $\mathbf{T}^*_m$ by $\boldsymbol{\tau}^*_i$, and the j-th column of $\mathbf{Q}'_m$ by $\mathbf{q}^*_j$, one gets

$$\begin{cases} \left[(\boldsymbol{\tau}^*_i)^T \,|\, \mathbf{1}\right] &= \left[\dfrac{1}{\sqrt{1\cdot 2}},\ldots,\dfrac{1}{\sqrt{(i-1)i}},\underbrace{\sqrt{\dfrac{i+1}{i}}}_{i\text{-th}},0,\ldots,0,1\right],\ i \le m; \\ \left[(\boldsymbol{\tau}^*_{m+1})^T \,|\, \mathbf{1}\right] &= [0,0,\ldots,0,1]\,; \end{cases}$$

$$\mathbf{q}^*_j = \begin{bmatrix} a_{j1}\sqrt{\dfrac{j}{j+1}} - \displaystyle\sum_{p=j+1}^{m} \dfrac{a_{p1}}{\sqrt{p(p+1)}} \\ \vdots \\ a_{jm}\sqrt{\dfrac{j}{j+1}} - \displaystyle\sum_{p=j+1}^{m} \dfrac{a_{pm}}{\sqrt{p(p+1)}} \\ b_j\sqrt{\dfrac{j}{j+1}} - \displaystyle\sum_{p=j+1}^{m} \dfrac{b_p}{\sqrt{p(p+1)}} \end{bmatrix},\ j = 1,2,\ldots,m.$$

Taking scalar product $s_{ij} = (\boldsymbol{\tau}^*_i)^T \cdot \mathbf{q}^*_j$, gives (4). □

Theorem 2.3. *If all elements of the matrices* $\mathbf{S}^{(k)} = [s^{(k)}_{ij}]^{m+1}_{i,j=1}$, $k \le n$, *are nonnegative, than the AIFS* $\Omega(\hat{\mathbf{T}}_m) = \{\hat{\mathbf{T}}_m; \mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \ldots, \mathbf{S}^{(n)}\}$ *has the convex hull property.*

Proof. For every $k = 1,2,\ldots,n$, and $i = 1,2,\ldots,m+1$, $\mathbf{S}^{(k)T}\mathbf{e}_i = [s^{(k)}_{ij}]^{m+1}_{j=1}$, i.e. the image of the i-th vertex of the simplex $\hat{\mathbf{T}}_m$ with the k-th linear transformation $\mathcal{L}_i : \mathbf{x} \mapsto \mathbf{S}^{(k)T}\mathbf{x}$, $\mathbf{x} \in \mathbb{V}^m$, is the point whose coordinates coincide with the i-th row of the row-stochastic matrix $\mathbf{S}^{(k)}$. Since all entries of the matrices $\mathbf{S}^{(k)}$ are nonnegative, the point $[s^{(k)}_{ij}]^{m+1}_{j=1}$ for $k = 1,2,\ldots,n$, $i = 1,2,\ldots,m+1$, belongs to the convex hull of $\hat{\mathbf{T}}_m$, and therefore $\mathbf{S}^{(k)T}\hat{\mathbf{T}}_m$ belongs to the convex hull of $\hat{\mathbf{T}}_m$, $k = 1,2,\ldots,n$. Now,

$$W_{\Omega(\hat{\mathbf{T}}_m)}(\hat{\mathbf{T}}_m) = \bigcup_{k=1}^{n} S^{(k)T}\hat{\mathbf{T}}_m \subseteq \mathrm{conv}\hat{\mathbf{T}}_m, \quad \text{and also,}$$

$$W^{\circ n}_{\Omega(\hat{\mathbf{T}}_m)}(\hat{\mathbf{T}}_m) \subseteq \mathrm{conv}W^{\circ(n-1)}_{\Omega(\hat{\mathbf{T}}_m)}(\hat{\mathbf{T}}_m) \subseteq \ldots \subseteq \mathrm{conv}W_{\Omega(\hat{\mathbf{T}}_m)}(\hat{\mathbf{T}}_m) \subseteq \mathrm{conv}\hat{\mathbf{T}}_m,$$

$$\text{i.e.}\quad \operatorname{att}(\Omega(\hat{\mathbf{T}}_m)) = \lim_{n\to\infty} W^{\circ n}_{\Omega(\hat{\mathbf{T}}_m)}(\hat{\mathbf{T}}_m) \subseteq \operatorname{conv}\hat{\mathbf{T}}_m.$$

□

As a corollary of the theorems 2.2 and 2.3, one can conclude that the sufficient condition for the convex hull property of the IFS $\{\mathbb{V}^m; w_1, w_2, \ldots, w_n\}$, where

$$w_k(\mathbf{x}) = \mathbf{A}(\mathbf{S}^{(k)})\mathbf{x} + \mathbf{b}(\mathbf{S}^{(k)}), \quad \mathbf{x} \in \mathbb{V}^m \quad k = 1, 2, \ldots, n,$$

($\mathbf{A}(\mathbf{S}^{(k)})$ and $\mathbf{b}(\mathbf{S}^{(k)})$ are obtained as in theorem 2.1), are

(5)
$$\begin{cases} \sqrt{\dfrac{j}{j+1}}\left[\displaystyle\sum_{q=1}^{i-1}\frac{a^{(k)}_{jq}}{\sqrt{q(q+1)}} + \sqrt{\frac{i+1}{i}}a^{(k)}_{ji} + b^{(k)}_j\right] - \\ -\displaystyle\sum_{p=j+1}^{m}\frac{1}{\sqrt{p(p+1)}}\left[\sum_{q=1}^{i-1}\frac{a^{(k)}_{pq}}{\sqrt{q(q+1)}} + \sqrt{\frac{i+1}{i}}a^{(k)}_{pi} + b^{(k)}_p\right] \geq 0,\ i,j \leq m, \\ \sqrt{\dfrac{j}{j+1}}b^{(k)}_j - \displaystyle\sum_{p=j+1}^{m}\frac{b^{(k)}_p}{\sqrt{p(p+1)}} \geq 0,\ j = 1, 2, \ldots, m. \end{cases}$$

3. Examples

Let's imagine a 4D tetrahedron (in fact a 4-simplex). It will be used to define the AIFS in the both examples that will follow. A regular tetrahedron $ABCD$ may serve as the base of a 4D – tetrahedron $ABCDE$ whose apex E is along the fourth dimension through the centre of $ABCD$. If E is so chosen that its distances from A, B, C, D are all equal to the edge AB, we have a regular simplex, which may be regarded in five ways as a tetrahedron, each vertex in turn serving as the apex while the remaining four form the base.

Example 1. In the first example the AIFS that generates the 4D Sierpinski pyramid will be considered, $\Omega(\hat{\mathbf{T}}_4) = \{\hat{\mathbf{T}}_4; \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_5\}$, $\hat{\mathbf{T}}_4 \subset \mathbb{V}^4$, $\hat{\mathbf{T}}_4$ is the standard 4-simplex, where

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix}; \quad \begin{aligned} \mathbf{S}_2 &= [\mathbf{e}_2\,\mathbf{e}_1\,\mathbf{e}_3\,\mathbf{e}_4\,\mathbf{e}_5]\mathbf{S}_1[\mathbf{e}_2\,\mathbf{e}_1\,\mathbf{e}_3\,\mathbf{e}_4\,\mathbf{e}_5]; \\ \mathbf{S}_3 &= [\mathbf{e}_3\,\mathbf{e}_2\,\mathbf{e}_1\,\mathbf{e}_4\,\mathbf{e}_5]\mathbf{S}_1[\mathbf{e}_3\,\mathbf{e}_2\,\mathbf{e}_1\,\mathbf{e}_4\,\mathbf{e}_5]; \\ \mathbf{S}_4 &= [\mathbf{e}_4\,\mathbf{e}_3\,\mathbf{e}_2\,\mathbf{e}_1\,\mathbf{e}_5]\mathbf{S}_1[\mathbf{e}_4\,\mathbf{e}_3\,\mathbf{e}_2\,\mathbf{e}_1\,\mathbf{e}_5]; \\ \mathbf{S}_5 &= [\mathbf{e}_5\,\mathbf{e}_4\,\mathbf{e}_3\,\mathbf{e}_2\,\mathbf{e}_1]\mathbf{S}_1[\mathbf{e}_5\,\mathbf{e}_4\,\mathbf{e}_3\,\mathbf{e}_2\,\mathbf{e}_1]. \end{aligned}$$

It can be noticed that this AIFS satisfies the conditions of the Theorem 2.3, therefore its attractor belongs to the convex hull of the simplex $\hat{\mathbf{T}}_4$, as it can be seen on the Figure 1, left.

The corresponding IFS is $\{\mathbb{V}^4; w_1, w_2, w_3, w_4, w_5\}$, $w_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x}+\mathbf{b}_i$, $i = 1, 2, 3, 4, 5$, where

$$\mathbf{A}_1 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} = \mathbf{A}_2 = \mathbf{A}_3 = \mathbf{A}_4 = \mathbf{A}_5;$$

$$\mathbf{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{6}} \\ \frac{1}{4\sqrt{3}} \\ \frac{\sqrt{5}}{4} \end{bmatrix}, \mathbf{b}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 2. Here, a non-fractal, continuous, four-dimensional attractor will be considered. We are speaking about 4D Bèzier curve, generated by the AIFS $\Omega(\hat{\mathbf{T}}_4)$ again endowed with the convex hull property. $\Omega(\hat{\mathbf{T}}_4) = \{\hat{\mathbf{T}}_4; \mathbf{S}_1, \mathbf{S}_2\}$, $\hat{\mathbf{T}}_4 \subset \mathbb{V}^4$, where

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \end{bmatrix}; \quad \mathbf{S}_2 = \begin{bmatrix} \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 2.1 explicitly gives the affine mappings,

$$w_1 : \mathbf{A}_1 = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{8\sqrt{2}} & -\frac{\sqrt{5}}{8\sqrt{2}} \\ -\frac{11}{16\sqrt{3}} & \frac{13}{48} & \frac{19}{48\sqrt{2}} & \frac{\sqrt{15}}{16\sqrt{2}} \\ -\frac{7}{8\sqrt{6}} & -\frac{7}{24\sqrt{2}} & \frac{5}{48} & \frac{\sqrt{15}}{16} \\ -\frac{\sqrt{5}}{8\sqrt{2}} & -\frac{\sqrt{5}}{8\sqrt{6}} & -\frac{\sqrt{5}}{16\sqrt{3}} & \frac{1}{16} \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{11}{8\sqrt{6}} \\ \frac{7}{8\sqrt{3}} \\ \frac{\sqrt{5}}{8} \end{bmatrix};$$

$$w_2 : \mathbf{A}_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{16\sqrt{2}} & -\frac{\sqrt{5}}{16\sqrt{2}} \\ -\frac{11}{16\sqrt{3}} & \frac{7}{48} & -\frac{1}{24\sqrt{2}} & -\frac{\sqrt{5}}{8\sqrt{6}} \\ \frac{7}{8\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{7}{96} & -\frac{5\sqrt{5}}{32\sqrt{3}} \\ \frac{\sqrt{5}}{8\sqrt{2}} & \frac{\sqrt{5}}{4\sqrt{6}} & \frac{7\sqrt{5}}{32\sqrt{3}} & \frac{7}{32} \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

that form the corresponding IFS $\{\mathbb{V}^4; w_1, w_2\}$. The attractor is shown on the Figure 1, right.

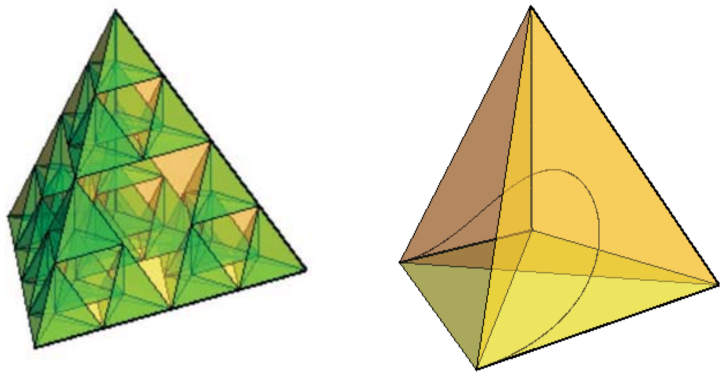

Figure 1. 4D Sierpinski pyramid and 4D Bèzier curve

4. Conclusion

Using a direct relationship between the orthonormal basis $\{\mathbf{e}_i\}$, in $\mathbb{R}^{m+1}$ and the m-dimensional hyperplane $\mathbb{V}^m \subset \mathbb{R}^{m+1}$ being the affine span of the points $\{\mathbf{e}_i\}$, the explicit formulae establishing bijective relation between IFS and AIFS are given. More concrete, the elements of the stochastic matrices of AIFS are given as functions of the items of the matrix and vector defining the affine transforms in IFS and vice versa. In addition, the sufficient condition for the AIFS to have convex hull property is established. Illustrative examples of two 4-dimensional fractal sets are given: a 4D Sierpinski pyramid and a 4D Bèzier curve.

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ПРИЛОЗИ КОН МОДЕЛИРАЊЕ НА ЧЕТИРИ-ДИМЕНЗИОНАЛНИ ФРАКТАЛИ

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Резиме

Афино инваријантните итеративни функциски системи се итеративни функциски системи чии афини трансформации се базираат на барицентрични координати во однос на одреден симплекс. Во овој труд разгледуваме 4-димензионални обопштувања на некои класични и некласични фрактални објекти, генерирани со помош на афино инваријантни итеративни функциски системи и покажуваме како овие системи можат да се користат за моделирање. Акцентот е ставен на едно од најпожелните својства во моделирањето, својството на конвексна обвивка.

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