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TWO RESULTS FOR ONE SPECIAL CLASS OF NOHOMOGENOUS VEKUA EQUATION

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Abstract. The Vekua equation (13) depending of its coefficients, defines different classes of so called generalized analytic functions. In this work two results for the nohomogenous Vekua equation (5) are given with the theorems 1 and 2. That is, with corresponding conditions, the general solution (9) i.e. (12) is found, of the mentioned equation (5).

1. INTRODUCTION

In the work [1], with the method of areolar series $W = \sum_{p,q=0}^{\infty} c_{pq} z^p \bar{z}^q$, is found a general solution of the basic Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = \lambda \overline{W} \tag{1}$$

with complex coefficient $\lambda \in \mathbb{C}$ in the next shape

$$\begin{gathered} W = \Phi(z) + \sum_{n=0}^{\infty} \frac{|\lambda|^{2n+2}}{(n+1)!} \bar{z}^{n+1} \iint \cdots \int \Phi(z) (dz)^{n+1} + \\ + \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}\lambda}{n!} z^n \iint \cdots \int \overline{\Phi}(z) (d\bar{z})^{n+1} \end{gathered} \tag{2}$$

The function $\Phi = \Phi(z)$ in (2) is an arbitrary analytic function from $z = x + iy$, in some considered area D from the complex plane $\mathbb{C}$, in the role of the integration constant.

Here we should mention that the expression

$$\frac{\hat{d}W}{d\bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{3}$$

in (1) is so called operator derivative of the complex function $W = W(z) = u(x,y) + iv(x,y)$ from $\bar{z} = x - iy$, introduced by Г.В. Колосов [2] in 1909. The operating rules for this derivative are formulated and proved in Г.Н. Положий monograph [3] page 18 - 31. In the mentioned monograph there is a definition of the so called operator integral from $\bar{z} = x - iy$

$$\overset{\wedge}{\int} f(z)\, d\bar{z} = F(z) + C \qquad \left(\frac{\hat{d}F}{d\bar{z}} = f(z) \quad in \quad D\right) \tag{4}$$

of the complex function $f = f(z)$ in the area $D \subseteq \mathbb{C}$, where its operating rules are proved, page 32 - 41.

In this work we are considering the nohomogenous Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = a\overline{W} + F(z) \tag{5}$$

where $a \in \mathbb{C}$ is given complex constant and $F = F(z)$ is given complex function from $z = x + iy \in D$.

Theorem 1. *If $F = F(z)$ is antyanalytic function from $z = x + iy$ i.e. function that is complex conjugated with analytic function from z (i.e. $\overline{F} = \overline{F}(z) = u - iv$ is analytic from $z \Leftrightarrow \frac{d\overline{F}}{d\bar{z}} = 0$ in D) than with the substitution*

$$\omega = \overline{a}W + \overline{F} \tag{6}$$

the nohomogenous Vekua equation (5) will be transformed in a basic Vekua equation with complex coefficient $\overline{a} \in \mathbb{C}$ i.e. in the equation

$$\frac{\hat{d}\omega}{d\bar{z}} = \bar{a}\bar{\omega}. \tag{7}$$

Proof. From the substitution (6), according to the operating rules for the operator derivative by $\overline{z} = x - iy$ (3), we get that

$$\frac{\hat{d}\omega}{d\bar{z}} = \bar{a}\frac{\hat{d}W}{d\bar{z}} + \frac{\hat{d}\overline{F}}{d\bar{z}} =$$

$$= \bar{a}\frac{\hat{d}W}{d\bar{z}} + 0 = \bar{a}\frac{\hat{d}W}{d\bar{z}},$$

and $\frac{\hat{d}W}{d\bar{z}} = \frac{1}{\bar{a}}\frac{\hat{d}\omega}{d\bar{z}}$.

Using the substitution (6) in the equation (5), previously written in the shape

$$\frac{\hat{d}W}{d\bar{z}} = \overline{\overline{a}W + \overline{F}(z)} \tag{5'}$$

we get that

$$\frac{1}{\bar{a}}\frac{\hat{d}\omega}{d\bar{z}} = \bar{\omega}$$

i.e.

$$\frac{\hat{d}\omega}{d\bar{z}} = \bar{a}\bar{\omega} \tag{7}$$

With this the theorem is proved. □

Corollary 1. *As (7) is a basic Vekua equation with complex coefficient i.e. it is an equation shaped as in (1) ($\lambda = \bar{a}$) according to the unknown function $\omega = \omega(z)$, than according to (2), the general solution of (7) is*

$$\begin{aligned}\omega = \Phi(z) + \sum_{n=0}^{\infty} \tfrac{|a|^{2n+2}}{(n+1)!} \bar{z}^{n+1} \iint \cdots \int \Phi(z)\,(dz)^{n+1} + \\ + \sum_{n=0}^{\infty} \tfrac{|a|^{2n}\bar{a}}{n!} z^{n} \iint \cdots \int \overline{\Phi}(z)\,(d\bar{z})^{n+1}.\end{aligned} \tag{8}$$

According to the substitution (6), the general solution of the nohomogenous equation (5) is given with the formula

$$\begin{aligned}W = \tfrac{1}{\bar{a}} \Bigg[\Phi(z) + \sum_{n=0}^{\infty} \tfrac{|a|^{2n+2}}{(n+1)!} \bar{z}^{n+1} \iint \cdots \int \Phi(z)\,(dz)^{n+1} + \\ + \sum_{n=0}^{\infty} \tfrac{|a|^{2n}\bar{a}}{n!} z^{n} \iint \cdots \int \overline{\Phi}(z)\,(d\bar{z})^{n+1} - \overline{F}(z)\Bigg].\end{aligned} \tag{9}$$

Theorem 2. *If $W_1 = W_1(z)$ is one solution of the nohomogenous Vekua equation (5), than with the linear substitution*

$$W = \omega + W_1 \tag{10}$$

of the unknown function $W = W(z)$ the equation (5) will be transformed in a basic Vecua equation with a constant coefficient according to the new unknown function $\omega = \omega(z)$ i.e. in the equation

$$\frac{\hat{d}\omega}{d\bar{z}} = a\bar{\omega} \tag{11}$$

Proof. Using the substitution (10) in the nohomogenous Vecua equation (5) we have

$$\frac{\hat{d}\omega}{d\bar{z}} + \frac{\hat{d}W_1}{d\bar{z}} = a\overline{(\omega + W_1)} + F$$

i.e.

$$\frac{\hat{d}\omega}{d\bar{z}} + \frac{\hat{d}W_1}{d\bar{z}} = a\overline{\omega} + a\overline{W}_1 + F,$$

and if we take into consideration that $W_1 = W_1(z)$ is a solution of (5) (i.e. $\frac{\hat{d}W_1}{d\bar{z}} = = a\overline{W}_1 + F,\ \forall z \in D$) we get that

$$\frac{\hat{d}\omega}{d\bar{z}} = a\bar{\omega}, \tag{11}$$

and that is what we wanted to prove. □

Corollary 2. *The equation (11) is from the shape as in (1) (i.e $\lambda = a \in \mathbb{C}$) for the unknown function $\omega = \omega(z)$. According to (2) its general solution is*

$$\begin{aligned}\omega = \Phi(z) + \sum_{n=0}^{\infty} \tfrac{|a|^{2n+2}}{(n+1)!} \bar{z}^{n+1} \iint \cdots \int \Phi(z)\,(dz)^{n+1} + \\ + \sum_{n=0}^{\infty} \tfrac{|a|^{2n}a}{n!} z^{n} \iint \cdots \int \overline{\Phi}(z)\,(d\bar{z})^{n+1},\end{aligned}$$

and according to the substitution (10) the general solution of the nohomogenous Vekua equation (5) is

$$W=\Phi(z)+\sum_{n=0}^{\infty}\frac{|a|^{2n+2}}{(n+1)!}\bar{z}^{n+1}\iint\cdots\int\Phi(z)\,(dz)^{n+1}+ \\ +\sum_{n=0}^{\infty}\frac{|a|^{2n}a}{n!}z^{n}\iint\cdots\int\overline{\Phi}(z)\,(d\bar{z})^{n+1}+W_1(z)\,. \tag{12}$$

Note 1: In the already mentioned work [1] it is shown that the general solution (2) of the basic Vekua equation with constant coefficient (1) can be written in one of the following simplified shapes

$$W=\Phi(z)+\sum_{n=0}^{\infty}\frac{|\lambda|^{2n+2}}{(n+1)!\cdot n!}\int\bar{z}^{n+1}(z-\zeta)^{n}\,\Phi(\zeta)\,d\zeta+ \\ +\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}\lambda}{(n!)^{2}}\int z^{n}\left(\bar{z}-\bar{\zeta}\right)^{n}\overline{\Phi}(\zeta)\,d\bar{\zeta} \tag{2'}$$

or

$$W=\Phi(z)+\int\overline{\frac{\hat{d}S}{d\bar{z}}}\Phi(\zeta)\,d\zeta+\lambda\int S\overline{\Phi}(\zeta)\,d\bar{\zeta} \tag{2''}$$

where

$$S=S(u)=\sum_{n=0}^{\infty}\frac{u^{n}}{(n!)^{2}},\quad u=|\lambda|^{2}z\left(\bar{z}-\bar{\zeta}\right).$$

Let's note that the function $S=S(u)$ is a solution of the ordinary differential equation of second order

$$uS''(u)+S'(u)-S(u)=0.$$

This means that the general solution (9) of the nohomogenous Vekua equation (5) from the theorem 1, in this work, can be written in one of the following shapes

$$W=\frac{1}{\bar{a}}\left[\Phi(z)+\sum_{n=0}^{\infty}\frac{|a|^{2n+2}}{(n+1)!\cdot n!}\int\bar{z}^{n+1}(z-\zeta)^{n}\,\Phi(\zeta)\,d\zeta+\right. \\ \left.+\sum_{n=0}^{\infty}\frac{|a|^{2n}\bar{a}}{(n!)^{2}}\int z^{n}\left(\bar{z}-\bar{\zeta}\right)^{n}\overline{\Phi}(\zeta)\,d\bar{\zeta}-\overline{F}(z)\right] \tag{9'}$$

or

$$W=\frac{1}{\bar{a}}\left[\Phi(z)+\int\overline{\frac{\hat{d}S}{d\bar{z}}}\Phi(\zeta)\,d\zeta+\bar{a}\int S\overline{\Phi}(\zeta)\,d\bar{\zeta}-\overline{F}(z)\right] \tag{9''}$$

where

$$S=S(u)=\sum_{n=0}^{\infty}\frac{u^{n}}{(n!)^{2}},\quad u=|a|^{2}z\left(\bar{z}-\bar{\zeta}\right).$$

Similar to this, the general solution (12) of the nohomogenous equation (5) from the theorem 2 in this work can be written in one of the following shapes

$$W=\Phi(z)+\sum_{n=0}^{\infty}\frac{|a|^{2n+2}}{(n+1)!\cdot n!}\int\bar{z}^{n+1}(z-\zeta)^{n}\,\Phi(\zeta)\,d\zeta+ \\ +\sum_{n=0}^{\infty}\frac{|a|^{2n}a}{(n!)^{2}}\int z^{n}\left(\bar{z}-\bar{\zeta}\right)^{n}\overline{\Phi}(\zeta)\,d\bar{\zeta}+W_1(z) \tag{12'}$$

or

$$W = \Phi(z) + \int \overline{\frac{\hat{d}S}{d\bar{z}}} \Phi(\zeta)\, d\zeta + a \int S\overline{\Phi}(\zeta)\, d\bar{\zeta} + W_1(z) \tag{12''}$$

where

$$S = S(u) = \sum_{n=0}^{\infty} \frac{u^n}{(n!)^2}, \quad u = |a|^2 z \left(\bar{z} - \bar{\zeta}\right).$$

Note 2: The nohomogenous Vekua equation (5) is a special case of the general Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \tag{13}$$

which depending of the coefficients $A = A(z), B = B(z)$, $F = F(z)$, $z \in D \subseteq \mathbb{C}$, defines different classes of so called generalized analytic functions. Here, we should mention that for the equations of the first order

$$F^*\left(z, W, \frac{\hat{d}W}{d\bar{z}}\right) = 0 \tag{14}$$

where F^* is an analytic function from the derivative $\frac{\hat{d}W}{d\bar{z}}$, but it is not analytic according to the unknown function $W = W(z)$ (to whom belongs the Vecua equation (13), too) doesn't exist quadrature methods for solving them. The generalized analytic functions $W = W(z)$, defined with the equations shaped as in (14) as well as the generalized analytic functions, defined with the equations of higher order

$$F\left(z, W, \frac{\hat{d}W}{d\bar{z}}, \frac{\hat{d}^2W}{d\bar{z}^2}, \cdots, \frac{\hat{d}^nW}{d\bar{z}^n}\right) = 0,$$

are subject for exploring of very big number of scientists around the world. This big interest in this problems can be seen in [4]. Now we will quote some of this works, which we have met and whose results are used in our work: [3], [5], [6], [7], [8], [9], [10], [11] and many others.

Note 3: This work is a generalization of the results from the work [12].

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