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AN EXAMPLE OF SEQUENCE OF BLASCHKE PRODUCTS WHICH CONVERGE UNIFORMLY AT A UNIT DISK

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Abstract

The question of existence of sequence of Blaschke products which uniformly converges on the whole unit disk $D = \{z: |z| < 1\}$ is an important one. So, in this paper it is given an example of sequence of Blaschke products which uniformly converges on the whole unit disk $D = \{z: |z| < 1\}$ and even more whose boundary value is Blaschke product.

The definition of Blaschke product and some of this properties are given in the following:

Theorem A ([2]). *Let $a_1, a_2, \dots$ be a sequence of complex numbers such that $0 < |a_1| \leq |a_2| \leq \cdots < 1$ and $\sum\limits_{n=1}^{\infty}(1 - |a_n|) < \infty$. Then the infinite product*

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

converges uniformly in each disk $|z| \leq R < 1$. Each an is zero of $B(z)$, with multiplicity equal to the number of times it occurs in the sequence; and $B(z)$ has no other zeros in $|z| < 1$. Finally $|B(z)| < 1$ in $|z| < 1$, and $|B(e^{iz})| = 1$ a.e.

The function $B(z)$, in the theorem above is called Blaschke product. We mention that for real numbers $a_1, a_2, \dots, a_n$ the form of the Blaschke product is

$$B(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \overline{a}_n z}.$$

Now, we will give the example.

Let

$$\lambda_k^{(m)} = r_k e^{-a_k^{(m)}}, \quad \text{where} \quad \sum_{k=1}^{\infty}(1 - r_k) < +\infty, \quad r_k \in \mathbf{R}$$

and

$$a_k^{(m)} = (1 - r_k)^3 \frac{1}{m}, \quad m \in \mathbf{N}.$$

We consider the Blaschke products

$$b_m(z) = \prod_{k=1}^{\infty} \frac{\lambda_k^{(m)} - z}{1 - \lambda_k^{(m)} z}, \quad m \in \mathbf{N}. \tag{1}$$

We will use Cauchy's criterion for uniformly convergence at sequences at functions to show uniform convergence on the unit disk for sequence of Blaschke products (1).

The next inequalities will be used in the subsequent work.

i) If $r \in (0,1)$ and $|z| < 1$, then $1 - r < |1 - rz|$.

ii) If $x > 0$ and $y > 0$ then $|e^{-x} - e^{-y}| \leq |x - y|$.

iii) Let $a_1, a_2, \dots, a_n$ and $c_1, c_2, \dots, c_n$ are complex numbers, all having module less than 1. Then the inequality

$$\left| \prod_{j=1}^{n} a_j - \prod_{j=1}^{n} c_j \right| \leq \sum_{j=1}^{n} |a_j - c_j|.$$

holds.

We will do the next estimations:

$$|b_m(z) - b_{m+p}(z)| =$$

$$= \left| \prod_{k=1}^{\infty} \frac{\lambda_k^{(m)} - z}{1 - \lambda_k^{(m)} z} - \prod_{k=1}^{\infty} \frac{\lambda_k^{(m+p)} - z}{1 - \lambda_k^{(m+p)} z} \right| \leq$$

$$\leq \sum_{k=1}^{\infty} \left| \frac{\lambda_k^{(m)} - z}{1-\lambda_k^{(m)} z} - \frac{\lambda_k^{(m+p)} - z}{1-\lambda_k^{(m+p)} z} \right| =$$

$$= \sum_{k=1}^{\infty} \left| \frac{\lambda_k^{(m)} - \lambda_k^{(m+p)} - (\lambda_k^{(m)} - \lambda_k^{(m+p)}) z^2}{(1-\lambda_k^{(m)} z)(1-\lambda_k^{(m+p)} z)} \right| =$$

$$= \sum_{k=1}^{\infty} \left| \frac{(\lambda_k^{(m)} - \lambda_k^{(m+p)})(1-z^2)}{(1-\lambda_k^{(m)} z)(1-\lambda_k^{(m+p)} z)} \right| \leq$$

$$\leq 2 \sum_{k=1}^{\infty} \frac{|\lambda_k^{(m)} - \lambda_k^{(m+p)}|}{|1-\lambda_k^{(m)} z|\,|1-\lambda_k^{(m+p)} z|} \leq$$

$$\leq 2 \sum_{k=1}^{\infty} \frac{|\lambda_k^{(m)} - \lambda_k^{(m+p)}|}{|1-\lambda_k^{(m)}|\,|1-\lambda_k^{(m+p)}|} =$$

$$= 2 \sum_{k=1}^{\infty} \frac{|r_k\, e^{-a_k^{(m)}} - r_k\, e^{-a_k^{(m+p)}}|}{|1-r\, e^{-a_k^{(m)}}|\,|1-r_k\, e^{-a_k^{(m+p)}}|} \leq$$

$$\leq 2 \sum_{k=1}^{\infty} \frac{|e^{-a_k^{(m)}} - e^{-a_k^{(m+p)}}|}{(1-r)^2} \leq$$

$$\leq 2 \sum_{k=1}^{\infty} \frac{r_k}{(1-r_k)^2} \, |a_k^{(m+p)} - a_k^{(m)}| =$$

$$= 2 \sum_{k=1}^{\infty} \frac{r_k}{(1-r_k)^2} \left| (1-r_k)^3 \left(\frac{1}{m+p} - \frac{1}{m} \right) \right| =$$

$$= \frac{2p}{m+p} \sum_{k=1}^{\infty} r_k (1-r_k)\,.$$

Taking into consideration previous estimations and because series $\sum\limits_{k=1}^{\infty} r_k(1-r_k)$ converges, (it is true $\sum\limits_{k=1}^{\infty} (1-r_k < \infty)$, we obtain that the sequence of Blaschke products (1) uniformly converge on the whole unit disk D, using Cauchy's criterion for uniformly convergence on the sequences of functions.

A question for the nature of the function $\lim_{n\to\infty} b_n(z)$ arises. Tumarkin in his paper proves the next criterion for the nature of the function which is limit of a sequence of Blaschke products which uniformly converge on the compacts in the unit disk.

Theorem B ([3]). *Let*

$$B_k(z) = z^{\beta_k} \prod_{j=1}^{\infty} \frac{\alpha_{k,j} - z}{1 - \overline{\alpha}_{k,j} z} \frac{|\alpha_{k,j}|}{\alpha_{k,j}}$$

is sequence of Blaschke products which uniformly converge on the compacts in the unit disk towards function $B(z)$. Necessary and sufficient condition for a function to be Blaschke product are:

1) *In every disk $\{z: |z| < r\}$, $0 < r < 1$, the number of zeros of the functions $(B_k(z))$ is uniformly bounded.*

2) *Given any positive number ε, there exists R, $0 < R < 1$, such that for all $k \in \mathbf{N}$ holds:*

$$\sum_{|\alpha_{k,j}|>R} (1 - |\alpha_{k,j}|) < \varepsilon .$$

We verify the conditions 1) and 2) for the sequences of Blaschke products (1).

If there exists $R \in (0,1)$ so that in $\{z: |z| < R\}$ the number of zeros of the sequence of Blaschke products (1) is not uniformly bounded, we can choose a sequence of integers (k_s), who is strictly increas sequence, and for some natural number m holds $r_{k_s} e^{-(1-r_{k_s})^3 \frac{1}{m}} < R$, ($m$ depends on k_s). Because $k_s \to \infty$ when $s \to \infty$ and $r_k \to 1$ when $k \to \infty$ we obtain contradiction. So the number of zeros of the sequence of Blaschke products (1) is uniformly bounded on each disk $\{z: |z| < R\}$, $R \in (0,1)$.

It remains to show that for the sequence of Blaschke products (1) condition 2) of the above criterion hold. For given $\varepsilon > 0$, the convergence of the $\sum_{k=1}^{\infty} (1 - \lambda_k^{(1)})$ implies that there is $p \in \mathbf{N}$ so that $\sum_{k=p}^{\infty} (1 - \lambda_k^{(1)}) < \varepsilon$. Let $R = r_p$. Now, because $\lambda_p^{(1)} \le \lambda_p^{(1)} < r_p$ for $m \in \mathbf{N}$, we obtain that $\sum_{\lambda_k^{(m)}>R} (1 - \lambda_k^{(m)}) < \varepsilon$ for all $m \in \mathbf{N}$.

This shows that the sequence of Blaschke products (1) converges towards function which is Blaschke product.

References

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ПРИМЕР НА НИЗА ОД БЛАШКЕОВИ ПРОИЗВОДИ КОЈА РАМНОМЕРНО КОНВЕРГИРА НА ЕДИНИЧНИОТ ДИСК

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Резиме

Во трудот е даден пример на низа од Блашкеови производи која рамномерно конвергира на целиот единичен диск $D = \{z : |z| < 1\}$. Покажано е дека низата од Блашкеови производи, дадена во трудот, конвергира кон функција која е Блашкеов производ.

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