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ON SOME QUADRATURES IN CERTAIN SPECIAL CLASSES OF THE VEKUA EQUATIONS

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Abstract. In the paper, two equations with constant coefficients (1) and (17) are being considered with the method of areolar series. The solutions found are cases where an arbitrary analytic functions $\phi=\phi(z)$ plays the role of an integrated constant. So, we get quadratures formulas for the equations of that kind. Notice that the equations (1) and (17) are special cases of the known Vekua equation, that help us to define different classes of generalised analytic functions studied by Vekua in the sense of iteration [1].

Let

$$\frac{\partial w}{\partial \bar{z}} = \lambda \bar{w}, \quad (\lambda \in \mathbb{C}) \tag{1}$$

be a given Vekua equation with unknown function $w=u(x,y)+iv(x,y)$. There $z=x+iy$ and

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{2}$$

is so called operational derivative on $\bar{z}=x-iy$ of the function w [2].

We look for the solution of (1) in the form of areolar series

$$w = \sum_{p,q=0}^{\infty} C_{pq} z^p \bar{z}^q \tag{3}$$

where C_{pq} are constants that will be determinated. The substitution of the function w, defined by (3), in the equation (1) we get the identity

$$\sum_{p=0,q=1}^{\infty} qC_{pq} z^p \bar{z}^{q-1} = \sum_{p,q=0}^{\infty} \lambda \bar{C}_{pq} \bar{z}^p z^q \tag{4}$$

If we equalize the coefficients in front of equal orders of $z^p\bar{z}^q$, and collect the members C_{pq} by the low $p+q=n$ $(n=0,1,2,\dots)$, we get an infinity system of equations

$$
\begin{aligned}
C_{01} &= \lambda\overline{C}_{00} \\
C_{11} &= \lambda\overline{C}_{01} \\
2C_{02} &= \lambda\overline{C}_{10} \\
C_{21} &= \lambda\overline{C}_{02} \\
2C_{12} &= \lambda\overline{C}_{11} \\
3C_{03} &= \lambda\overline{C}_{20} \\
&\vdots \\
C_{n1} &= \lambda\overline{C}_{on} \\
2C_{n-1\;2} &= \lambda\overline{C}_{1\;n-1} \\
3C_{n-2\;3} &= \lambda\overline{C}_{2\;n-2} \qquad\qquad (n=0,1,2,\ldots) \\
&\vdots \\
(n+1)C_{o\,n+1} &= \lambda\overline{C}_{n\,o} \\
&\vdots
\end{aligned}
\tag{5}
$$

The infinity system of equations provides the coefficients C_{pq} $(q\neq 0)$ that are dependent of C_{po} $(p=0,1,2,\ldots)$. The computation is given by the following schema

$$
\begin{array}{cccccccccccccccc}
C_{00} & \to & C_{01} & \to & C_{11} & \to & C_{12} & \to & C_{21} & \to & C_{22} & \to & C_{23} & \to & C_{33} & \to \cdots \\
C_{10} & \to & C_{02} & \to & C_{21} & \to & C_{13} & \to & C_{32} & \to & C_{24} & \to & C_{43} & \to & C_{35} & \to \cdots \\
C_{20} & \to & C_{03} & \to & C_{31} & \to & C_{14} & \to & C_{42} & \to & C_{25} & \to & C_{53} & \to & C_{36} & \to \cdots \\
C_{30} & \to & C_{04} & \to & C_{41} & \to & C_{15} & \to & C_{52} & \to & C_{26} & \to & C_{63} & \to & C_{37} & \to \cdots \\
C_{40} & \to & C_{05} & \to & C_{51} & \to & C_{16} & \to & C_{62} & \to & C_{27} & \to & C_{73} & \to & C_{38} & \to \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Then for n=1,2,... the formulas are obtained as

$$
\begin{aligned}
C_{o\;n} &= \frac{1}{n}\lambda\overline{C}_{n-1\;o} \\
C_{n\;1} &= \frac{1}{1}\lambda\overline{C}_{o\;n} = \frac{1}{n}|\lambda|^2 C_{n-1\;o} \\
C_{1\;n+1} &= \frac{1}{n+1}\lambda\overline{C}_{n\,1} = \frac{1}{n(n+1)}|\lambda|^2\lambda\overline{C}_{n-1\;o} \\
C_{n+1\;2} &= \frac{1}{2}\lambda\overline{C}_{1\;n+1} = \frac{1}{2n(n+1)}|\lambda|^4 C_{n-1\;o}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
C_{2\ n+2} &= \frac{1}{n+2}\lambda\overline{C}_{n+1\ 2} = \frac{1}{2n(n+1)(n+2)}|\lambda|^4\lambda\overline{C}_{n-1\ o} \\
C_{n+2\ 3} &= \frac{1}{3}\lambda\overline{C}_{2\ n+2} = \frac{1}{2\cdot 3\cdot n(n+1)(n+2)}|\lambda|^6 C_{n-1\ o} \\
C_{3\ n+3} &= \frac{1}{n+3}\lambda\overline{C}_{n+2\ 3} = \frac{1}{2\cdot 3\cdot n(n+1)(n+2)(n+3)}|\lambda|^6\lambda\overline{C}_{n-1\ o} \\
C_{n+3\ 4} &= \frac{1}{4}\lambda\overline{C}_{3\ n+3} = \frac{1}{2\cdot 3\cdot 4\cdot n(n+1)(n+2)(n+3)}|\lambda|^8 C_{n-1\ o} \\
C_{4\ n+4} &= \frac{1}{n+4}\lambda\overline{C}_{n+3\ 4} = \frac{1}{2\cdot 3\cdot 4\cdot n(n+1)(n+2)(n+3)(n+4)}|\lambda|^8\lambda\overline{C}_{n-1\ o} \\
&\vdots
\end{aligned}
\tag{6}
$$

If the coefficients C_{pq} $(q\neq 0)$ in (3) expressed with the arbitrary coefficients C_{po} $(p=0,1,2,\ldots)$, by formulas (6), then the solution to the equation (1)

$$
\begin{aligned}
W(z,\overline{z}) = {} & C_{oo} \\
& + C_{1o}z + \lambda\overline{C}_{oo}\overline{z} \\
& + C_{2o}z^2 + |\lambda|^2 C_{oo}z\overline{z} + \frac{\lambda}{2}\overline{C}_{1o}\overline{z}^2 \\
& + C_{3o}z^3 + \frac{|\lambda|^2}{2}C_{1o}z^2\overline{z} + \frac{|\lambda|^2\lambda}{2}\overline{C}_{oo}z\overline{z}^2 + \frac{\lambda}{3}\overline{C}_{2o}\overline{z}^3 \\
& + C_{4o}z^4 + \frac{|\lambda|^2}{1\cdot 3}C_{2o}z^3\overline{z} + \frac{|\lambda|^4}{2^2}C_{oo}z^2\overline{z}^2 + \frac{|\lambda|^2\lambda}{2\cdot 3}\overline{C}_{1o}z\overline{z}^3 + \frac{\lambda}{4}\overline{C}_{3o}\overline{z}^4 \\
& + C_{5o}z^5 + \frac{|\lambda|^2}{4}C_{3o}z^4\overline{z} + \frac{|\lambda|^4}{2^2\cdot 3}C_{1o}z^3\overline{z}^2 + \frac{|\lambda|^4\lambda}{2^2\cdot 3}\overline{C}_{oo}z^2\overline{z}^3 + \frac{|\lambda|^2\lambda}{3\cdot 4}\overline{C}_{2o}z\overline{z}^4 + \frac{\lambda}{5}\overline{C}_{4o}\overline{z}^5 \\
& + C_{6o}z^6 + \frac{|\lambda|^2}{5}C_{4o}z^5\overline{z} + \frac{|\lambda|^4}{2\cdot 3\cdot 4}C_{2o}z^4\overline{z}^2 + \frac{|\lambda|^6}{2^2\cdot 3^2}C_{oo}z^3\overline{z}^3 + \frac{|\lambda|^4\lambda}{2^2\cdot 3\cdot 4}\overline{C}_{1o}z^2\overline{z}^4 + \frac{|\lambda|^2\lambda}{4\cdot 5}\overline{C}_{3o}z\overline{z}^5 \\
& + \\
& \vdots
\end{aligned}
\tag{7}
$$

In simillar manner, the classification of the members of w as in (7), produces

$$
\begin{aligned}
W = {} & \sum_{p=o}^{\infty} C_{po}z^p + \lambda\sum_{p=o}^{\infty}\overline{C}_{po}\frac{\overline{z}^{p+1}}{p+1} + \\
& + |\lambda|^2\overline{z}\sum_{p=o}^{\infty} C_{po}\frac{z^{p+1}}{p+1} + |\lambda|^2\lambda z\sum_{p=o}^{\infty}\overline{C}_{po}\frac{\overline{z}^{p+2}}{(p+1)(p+2)} + \\
& + |\lambda|^4\frac{\overline{z}^2}{2!}\sum_{p=o}^{\infty} C_{po}\frac{z^{p+2}}{(p+1)(p+2)} + |\lambda|^4\lambda\frac{z^2}{2!}\sum_{p=o}^{\infty}\overline{C}_{po}\frac{\overline{z}^{p+3}}{(p+1)(p+2)(p+3)} + \\
& + |\lambda|^6\frac{\overline{z}^3}{3!}\sum_{p=o}^{\infty} C_{po}\frac{z^{p+3}}{(p+1)(p+2)(p+3)} + |\lambda|^6\lambda\frac{z^3}{3!}\sum_{p=o}^{\infty}\overline{C}_{po}\frac{\overline{z}^{p+4}}{(p+1)(p+2)(p+3)(p+4)} \\
& + \\
& \vdots
\end{aligned}
\tag{8}
$$

The coefficients C_{po} $(p=0,1,2,...)$ are arbitrary, but such that the series

$$\sum_{p=o}^{\infty} C_{po} z^p$$

converges. In the disc of converges

$$D = \{z: |z| < R, \quad R \neq 0\}$$

the series defines arbitrary analytic function $\phi(z)$:

$$\sum_{p=o}^{\infty} C_{po} z^p = \phi(z), \quad z \in D.$$

For $z \in D$, the formula (8) could be writen in as

$$\begin{aligned} W = {} & \phi(Z) + \lambda\int\overline{\phi}(z)d\overline{z} + \\ & + |\lambda|^2\overline{z}\int\phi(z)dz + |\lambda|^2\lambda z\int\int\overline{\phi}(z)(d\overline{z})^2 + \\ & + |\lambda|^4 \frac{\overline{z}^2}{2!}\int\int\phi(z)(dz)^2 + |\lambda|^4\lambda\frac{z^2}{2!}\int\int\int\overline{\phi}(z)(d\overline{z})^3 + \\ & + |\lambda|^6 \frac{\overline{z}^3}{3!}\int\int\int\phi(z)(dz)^3 + |\lambda|^6\lambda\frac{z^3}{3!}\int\int\int\int\overline{\phi}(z)(d\overline{z})^4 + \\ & + \\ & \vdots \end{aligned}$$

or in brief

$$\begin{aligned} W = \sum_{n=o}^{\infty} \frac{|\lambda|^{2n}}{n!}\Big[& \overline{z}^n\int\int\cdots\int\phi(z)(dz)^n + \\ & + \lambda z^n\int\int\cdots\int\int\overline{\phi}(z)(d\overline{z})^{n+1}\Big]. \end{aligned} \tag{9}$$

Since $\phi=\phi(z)$ is an analytic function, each of the integrals

$$\int\int\cdots\int\phi(z)(dz)^n \text{ and } \int\int\cdots\int\int\phi(z)(dz)^{n+1}$$

is an analytic function in D, and therefore it is bounded in each closed disc

$$D_1 = \{z: |z| \leq \rho < R\}$$

$$\left|\int\int\cdots\int\phi(z)(dz)^n\right| < M, \qquad \left|\int\int\cdots\int\int\phi(z)(dz)^{n+1}\right| < N.$$

The use of the estimates

$$\left|\frac{|\lambda|^{2n}}{n!}\overline{z}^n\int\int\cdots\int\phi(z)(dz)^n\right| = \frac{|\lambda|^{2n}}{n!}|z|^n\left|\int\int\cdots\int\phi(z)(dz)^n\right| <$$

$$< \frac{|\lambda|^{2n}}{n!}R^n M$$

$$\left|\frac{|\lambda|^{2n}\lambda z^n}{n!}\iint\cdots\iint\overline{\phi}(z)\,(d\overline{z})^{n+1}\right| = \frac{|\lambda|^{2n+1}|z|^n}{n!}\left|\overline{\iint\cdots\iint\phi(z)\,(dz)^{n+1}}\right| < $$
$$< \frac{|\lambda|^{2n+1}R^n}{n!}N$$

and the convergence of the series

$$M\sum_{n=o}^{\infty}\frac{|\lambda|^{2n}R^n}{n!}, \qquad N\sum_{n=o}\frac{|\lambda|^{2n+1}R^n}{n!}N$$

yields the convergence of the series

$$\sum_{n=o}^{\infty}\frac{|\lambda|^{2n}}{n!}\,\overline{z}^n\iint\cdots\int\phi(z)\,(dz)^n$$

and

$$\sum_{n=o}^{\infty}\frac{|\lambda|^{2n}}{n!}\,z^n\iint\cdots\iint\overline{\phi}(z)\,(d\overline{z})^{n+1}$$

Therefore, the series in (9) converges for all $z\in D$ as a sum of two convergent series.

Theorem 1. The special equation of Vekua with constant coefficients (1) has a general solution (9), where $\phi=\phi(z)$ is an analytic function and plays the role of an integrated constant.

Let as write the solution of (9) in condensed form.

If in the general solution (9) one writes $\phi(z)$ as a separate addend and preserves it in the form

$$W = \phi(z) + \sum_{n=o}^{\infty}\frac{|\lambda|^{2n+2}}{(n+1)!}\,\overline{z}^{n+1}\iint\cdots\int\phi(z)\,(dz)^{n+1} + \tag{10}$$
$$+ \sum_{n=o}^{\infty}\frac{|\lambda|^{2n}\lambda}{n!}\,z^n\iint\cdots\iint\overline{\phi}(z)\,(d\overline{z})^{n+1}$$

by using the formula

$$\iint\cdots\int\phi(z)\,(dz)^k = \frac{1}{(k-1)!}\int(z-\zeta)^{k-1}\phi(\zeta)\,d\zeta, \quad \int\phi(z)\,dz \overset{\text{def}}{=\!=} \int\phi(\zeta)\,d\zeta$$

we get

$$W = \phi(z) + \sum_{n=o}^{\infty}\frac{|\lambda|^{2n+2}}{(n+1)!\cdot n!}\int\overline{z}^{n+1}(z-\zeta)^n\phi(\zeta)\,d\zeta + \tag{11}$$
$$+ \sum_{n=o}^{\infty}\frac{|\lambda|^{2n}\lambda}{(n!)^2}\int z^n(\overline{z}-\overline{\zeta})^n\overline{\phi}(\zeta)\,d\overline{\zeta}\,.$$

If in (11) the signums and integers change their places and it we put

$$\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} = S(u), \quad u=|\lambda|^2 z(\bar{z}-\bar{\zeta}) \tag{12}$$

we can write the solution of (1) in condesate form

$$W = \phi(z) + \int \frac{\partial S}{\partial \bar{z}} \phi(\zeta) d\zeta + \lambda \int S\bar{\phi}(\zeta) d\bar{\zeta} \tag{13}$$

We notice that the function $S=S(u)$, defined by (12), is a solution of the ordinary differential equation

$$uS''(u) + S'(u) - S(u) = 0, \tag{14}$$

which according to Kamke [3, page 401], has a solution

$$S = S(u) = S(|\lambda|^2 z(\bar{z}-\bar{\zeta})) = \mathfrak{Z}_o(2i\sqrt{|\lambda|^2 z(\bar{z}-\bar{\zeta})}).$$

$\mathfrak{Z}_o$ is so called cylindrical function of order zero.

Hence, the function (13) coned be written in the form

$$\begin{aligned} W = \phi(z) &+ \int \overline{\frac{\partial}{\partial \bar{z}} \mathfrak{Z}_o(2i\sqrt{|\lambda|^2 z(\bar{z}-\bar{\zeta})})\phi(\zeta) d\zeta} + \\ &+ \lambda \int \mathfrak{Z}_o(2i\sqrt{|\lambda|^2 z(\bar{z}-\bar{\zeta})})\bar{\phi}(\zeta) d\bar{\zeta} \end{aligned} \tag{15}$$

Theorem 2. The general solution (9) of the equation (1) can be expressed by one sided integrals of zero order cylindrical function (15).

Remark 1. The forms (9), (13) and (15) for the general solution of the equation (1), show that the unknown function W does not depend linearly on the integrated constant - the analytical function $\phi(z)$, which is property of every linear differential equation. That means that the equation (1) is transcedental. This is a result of the complex-conjugate operation on unknown function W.

Remark 2. If we put $W=u+iv$, $\lambda=a+ib$, the equation (1) becomes a complex form of the system of partial equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 2(au-bv) \\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} = 2(bu+av) \end{aligned} \tag{16}$$

Therefore, the system of equations (16) has a solution

$$u = \mathrm{Re}W$$
$$v = \mathrm{Im}W,$$

where w is given by one of (9), (13) or (15).

Theorem 3. The non homogenaes Vekua equation

$$\frac{\partial W}{\partial \bar{z}} = \lambda \overline{W} + \mu, \quad (\lambda, \mu \in \mathbb{C}) \tag{17}$$

with change of the variable

$$\bar{\lambda} W + \bar{\mu} = \omega \tag{18}$$

is transformed in an equation of form (1) with new unknown function $\omega=\omega(z)$.

Proof. The transformation (18) implies

$$\frac{\partial W}{\partial \bar{z}} = \frac{1}{\bar{\lambda}} \frac{\partial \omega}{\partial \bar{z}} \tag{19}$$

Since the equation (17) can be write in the form

$$\frac{\partial W}{\partial \bar{z}} = \overline{\bar{\lambda} W + \bar{\mu}},$$

the transformation (18) and the equation (19) gives

$$\frac{\partial \omega}{\partial \bar{z}} = \bar{\lambda}\, \bar{\omega} \tag{20}$$

Consequence. The equation (20) is of the form (1). It coned be solved by either one of the quadratures (9), (13) or (15), where w and λ are repland by ω and $\bar{\lambda}$ respectivety. Since, the transformation (18) implies

$$W = \frac{1}{\bar{\lambda}}\omega - \left(\frac{\bar{\mu}}{\bar{\lambda}}\right)$$

we get quadratures formulas for the equation (17).

For example, if we used the quadrature formula of the form (9) for solution of the equation (20), then we get the following quadrature formula

$$W = \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} \left[\frac{\bar{z}^n}{\bar{\lambda}} \iint \cdots \int \phi(z)\,(dz)^n + \right.$$

$$\left. + z^n \iint \cdots \iint \bar{\phi}(z)\,(d\bar{z})^{n+1} \right] - \left(\frac{\bar{\mu}}{\bar{\lambda}}\right)$$

for the equation (17).

R E F E R E N C E S

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ЗА НЕКОИ КВАДРАТУРИ ВО ДВЕ СПЕЦИЈАЛНИ КЛАСИ РАВЕНКИ ВЕКУА

Борко Илиевски

Р е з и м е

Во оваа работа, со метода на ареоларни редови, се разгледувани две ареоларни равенки со константни коефициенти (1) и (17). Најдени се нивни решенија во кои што е одделена произволна аналитична функција $\phi=\phi(z)$ во улога на интеграциона константа. Според тоа, имаме еден вид квадратурни формули за споменатите равенки. Да забележиме дека равенките (1) и (17) се специјални случаи на познатата равенка И.Н.Векуа, со чија што помош се дефинираат различни класи обопштени аналитични функции, третирана од самиот Векуа во смисла на итерации во трудот [1].

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