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BILINEAR HILBERT TRANSFORM IN $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$

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Abstract

Bilinear Hilbert transform in appropriate Colombeau algebras $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$ are given. The relations of bilinear Hilbert transforms with the corresponding definitions in Colombeau algebras are given through special regularizations.

Introduction and Preliminaries

The Hilbert transform (HT), as a simplest singular integral and a prototype of a pseudodifferencial operator, is used in many branches of analysis. We refer to the monograph [11] for the study of HT of Schwartz distributions and its subspaces.

Recently, bilinear Hilbert transform (BHT) on L^p spaces became interesting. Important papers of Lacey and Thiele [6]-[8] strongly stimulate the investigations in this direction. In [3] is given the extension of BHT on distributions.

Problem of multiplication in distribution spaces and the inconvenience of distribution spaces for nonlinear problems, motivate Colombeau to construct an algebra of generalized functions in which the multiplication of smooth function is the usual multiplication and in which the distribution spaces can be embedded and then multiplied. For this theory we refer to [4,5], [2], [9,10] and [12]. Distributions are embedded into Colombeau algebra under a proces of regularizations using special delta nets.

In this paper we are interested in the definition of BHT in Colombeau type algebras. For this reason, we consider weighted Colombeau algebras $\mathcal{G}_{L^p}$. With suitable regularizations we analyse the relations between the embedded BHT of distributions and the BHT of embedded distributions.

Breafly: we define the BHT in Colombeau's algebra $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$. We use the results of Lacey and Thiele in [6]-[8] and Bučkovska, Pilipović in [3] for definition and properties of bilinear HT. Then we prove the association of $H_\alpha(f * \delta_\varepsilon, a * \delta_\varepsilon)$ and $H^*_\alpha(f,a) * \delta_\varepsilon$.

In the sequel for $1 < p < \infty$ we will write $q = \dfrac{p}{p-1}$ and denote by $\mathcal{D}'_{L^p}$ the dual of the space $\mathcal{D}_{L^q}$. We will not recall the definitions (cf. [14], [11]).

2. Hilbert transform in Colombeau algebra $\mathcal{G}_{L^p}$

We refer to monographs given in the literature for the definitions and properties of general Colombeau type algebras. Here we will recall definitions of those algebras needed in the sequal. They are introduced by Biagioni in [2] and Oberguggenberger in [9].

Let Ω be an open set of $\mathbf{R}^n$. Denote by $\mathcal{E}[\Omega]$ the set of nets $(F_\varepsilon)_\varepsilon$, $\varepsilon \in (0,1)$, of smooth functions on Ω.

Definition 2.1 *The set $\mathcal{E}_{L^p}[\Omega]$ consists of all nets $(F_\varepsilon)_\varepsilon \in \mathcal{E}[\Omega]$ with the following property:*

For every $\alpha \in \mathbf{N}_0^n$ there exist $a > 0$ and $c > 0$ such that

$$||F_\varepsilon{}^{(\alpha)}||_p \leq c \cdot \varepsilon^{-a}.$$

The set of all L^p - null functions $\mathcal{N}_{L^p}[\Omega]$ consists of all nets $(F_\varepsilon)_\varepsilon \in \mathcal{E}_{L^p}[\Omega]$ such that for every $\alpha \in \mathbf{N}_0^n$ and every $a > 0$ there exist $c > 0$ such that

$$||F_\varepsilon{}^{(\alpha)}||_p \leq c \cdot \varepsilon^{a}.$$

For the sake of simplicity, we will write F_ε instead of $(F_\varepsilon)_\varepsilon$.

Theorems 2.1 and 2.2 which are to follow, are proved by Biagioni and Oberguggenberger.

Theorem 2.1 *$\mathcal{E}_{L^p}[\Omega]$ and $\mathcal{N}_{L^p}[\Omega]$ are vector spaces and algebras under the pointwise multiplication of representatives. Moreover with the differentiation defined by the differentiation of the representatives:*

$$[F_\varepsilon]^{(\alpha)} = [F_\varepsilon^{(\alpha)}],\ \alpha \in \mathbf{N}_0^n,$$

they are differential algebras. $\mathcal{N}_{L^p}[\Omega]$ is ideal of $\mathcal{E}_{L^p}[\Omega]$.

Definition 2.2 *The space of generalized functions on Ω, $\mathcal{G}_{L^p}(\Omega)$ is defined by*

$$\mathcal{G}_{L^p}(\Omega) = \mathcal{E}_{L^p}[\Omega]/\mathcal{N}_{L^p}[\Omega].$$

Elements of $\mathcal{G}_{L^p}$ will be denoted by $f = [F_\varepsilon]$, $g = [G_\varepsilon], \ldots$

Theorem 2.2 *$\mathcal{G}_{L^p}(\Omega)$ is an algebra under multiplication $[F_\varepsilon] \cdot [G_\varepsilon] = [F_\varepsilon \cdot G_\varepsilon]$.*

Definition 2.3 *Let $f = [F_\varepsilon]$, $g = [G_\varepsilon] \in \mathcal{D}'_{L^p}$. It is said that they are associated if*

$$\int (F_\varepsilon - G_\varepsilon)\psi \to 0$$

as $\varepsilon \to 0$, $\psi \in \mathcal{D}_{L^q}$.

2.1. Embeddings

The space $\mathcal{D}'_{L^p}$ is embedded into $\mathcal{G}_{L^p}$ by the use of a special net of mollifiers, which is a delta net, but constructed via a function with special properties. We will denote it δ_ε.

Let $\phi \in \mathcal{S}$, $\int \phi = 1$, $\int \phi x^k = 0$, for $k \geq 1$. In the sequel we will put

$$\delta_\varepsilon = \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon}),\ \varepsilon \in (0,1).$$

Regularizations of elements in $\mathcal{D}'_{L^p}$ by this delta net will lead to embeddings.

Let $f \in \mathcal{D}'_{L^p}$. Then the corresponding element in $\mathcal{G}_{L^p}$ is denoted by $Cd\, f$ and it is defined as

$$Cd\, f = [f * \delta_\varepsilon].$$

Then

$$H(Cd\, f) = [H(f * \delta_\varepsilon)] = \left[\left(f * \delta_\varepsilon\right) * p.v.\frac{1}{x}\right].$$

3. Bilinear Hilbert transform in $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$

Definition 3.1 *Let $F_\varepsilon \in \mathcal{E}_{L^{p_1}}$ and $A_\varepsilon \in \mathcal{E}_{L^{p_2}}$. Then*

$$H_{\alpha,A_\varepsilon}(F_\varepsilon) = p.v. \int F_\varepsilon(x-t)A_\varepsilon(x+\alpha t)\frac{dt}{t}, \quad \alpha \in \mathbb{R} \setminus \{-1, 0\}.$$

We have proved in [3] that if $a \in \mathcal{D}_{L^{p_2}}$ and $\varphi \in \mathcal{D}_{L^{p_1}}$ then

$$||(H_{\alpha,a}\varphi)^{(m)}||_p \leq C \sum_{k=0}^{m} \binom{m}{k} ||\varphi^{(m-k)}||_{p_1} \cdot ||a^{(k)}||_{p_2} \quad \forall m \in \mathbb{N}. \qquad (1)$$

Theorem 3.1 (i) *Let* $F_\varepsilon \in \mathcal{E}_{L^{p_1}}$ *and* $A_\varepsilon \in \mathcal{E}_{L^{p_2}}$. *Then* $H_{\alpha,A_\varepsilon}(F_\varepsilon) \in \mathcal{E}_{L^p}$.

(ii) *Let* $R_\varepsilon \in \mathcal{N}_{L^{p_1}}$ *and* $A_\varepsilon \in \mathcal{E}_{L^{p_2}}$. *Then* $H_{\alpha,A_\varepsilon}(R_\varepsilon) \in \mathcal{N}_{L^p}$.

(iii) *Let* $A_\varepsilon \in \mathcal{N}_{L^{p_2}}$ *and* $F_\varepsilon \in \mathcal{E}_{L^{p_1}}$. *Then* $H_{\alpha,A_\varepsilon}(F_\varepsilon) \in \mathcal{N}_{L^p}$.

Proof: (i) We have to prove that for every $m \in \mathbf{N}_0^n$, there exist $b \in \mathbf{R}$ and $A > 0$ such that:

$$\left\|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right\|_p \leq A \cdot \varepsilon^{-b}, \quad \varepsilon \in (0,1).$$

Using (1) we obtain

$$\begin{aligned}\left\|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right\|_p &\leq C \sum_{k=0}^{m} \binom{m}{k} \|F_\varepsilon^{(m-k)}\|_{p_1} \cdot \|A_\varepsilon^{(k)}\|_{p_2} \\ &\leq C \sum_{k=0}^{m} \binom{m}{k} c_1 \cdot \varepsilon^{-a_1} \cdot c_2 \cdot \varepsilon^{-a_2} \\ &\leq A \cdot \varepsilon^{-b}.\end{aligned}$$

(ii) Similar to the proof of (i), we have to prove that for every $m \in \mathbf{N}_0^n$, and for every $b \in \mathbf{R}$ there exist $B > 0$ such that:

$$\left\|\left(H_{\alpha,A_\varepsilon}(R_\varepsilon)\right)^{(m)}\right\|_p \leq B \cdot \varepsilon^{b}, \quad \varepsilon \in (0,1).$$

Using (1) we obtain

$$\begin{aligned}\left\|\left(H_{\alpha,A_\varepsilon}(R_\varepsilon)\right)^{(m)}\right\|_p &\leq C \sum_{k=0}^{m} \binom{m}{k} \|R_\varepsilon^{(m-k)}\|_{p_1} \cdot \|A_\varepsilon^{(k)}\|_{p_2} \\ &\leq C \sum_{k=0}^{m} \binom{m}{k} c_1 \cdot \varepsilon^{a_1} \cdot c_2 \cdot \varepsilon^{-a_2} \\ &\leq B \cdot \varepsilon^{b}.\end{aligned}$$

(iii) Now we have to prove that for every $m \in \mathbf{N}_0^n$, and for every $b \in \mathbf{R}$ there exist $B > 0$ such that:

$$\left\|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right\|_p \leq B \cdot \varepsilon^{b}, \quad \varepsilon \in (0,1).$$

Using (1) we have

$$\begin{aligned}\left\| \left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)} \right\|_p \leq & C \sum_{k=0}^{m} \binom{m}{k} ||F_\varepsilon^{(m-k)}||_{p_1} \cdot ||A_\varepsilon^{(k)}||_{p_2} \\ \leq & C \sum_{k=0}^{m} \binom{m}{k} c_1 \cdot \varepsilon^{-a_1} \cdot c_2 \cdot \varepsilon^{a_2} \\ \leq & B \cdot \varepsilon^{b}.\end{aligned}$$

Now we can define Bilinear Hilbert transform in $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$.

Definition 3.2 *Let $f \in \mathcal{G}_{L^{p_1}}$ and $a \in \mathcal{G}_{L^{p_2}}$. Then*

$$H_\alpha(f, a) = \left[H_\alpha(F_\varepsilon, A_\varepsilon)\right].$$

In the Theorem that follows we'll use the same notations as in section 2.1.

Theorem 3.2 *Let $f \in \mathcal{D}'_{L^{q_1}}$, $a \in \mathcal{D}_{L^{p_2}}$. Then $Cd\ H^*_{\alpha,a} f$ is assoociated with $H_{\alpha,a}(Cd\ f)$.*

Proof: Let $f \in \mathcal{D}'_{L^{q_1}}$, and $a \in \mathcal{D}_{L^{p_2}}$. Then, for every $\psi \in \mathcal{D}_{L^{p_1}}$ using Th.3 (ii) in [3] we get

$$\begin{aligned}\langle (H^*_{\alpha,a} f) * \delta_\varepsilon, \psi \rangle = & \langle H^*_{\alpha,a} f, \psi * \delta_\varepsilon \rangle \\ = & \langle f, H_{\alpha,a}(\psi * \delta_\varepsilon) \rangle.\end{aligned}$$

Then

$$\lim_{\varepsilon \to 0} \langle f, H_{\alpha,a}(\psi * \delta_\varepsilon) \rangle = \langle H^*_{\alpha,a} f, \psi \rangle.$$

From the other side we have:

$$\begin{aligned}\langle H_{\alpha,a}(f * \delta_\varepsilon), \psi \rangle = & \langle -H^*_{-1-\alpha,a}(f * \delta_\varepsilon), \psi \rangle \\ = & \langle f * \delta_\varepsilon, -H_{-1-\alpha,a} \psi \rangle = \langle f, \left(-H_{-1-\alpha} \psi\right) * \delta_\varepsilon \rangle,\end{aligned}$$

and if $\varepsilon \to 0$ we have

$$\lim_{\varepsilon \to 0} \langle f, -H_{-1-\alpha,a} \psi \rangle = \langle -H^*_{-1-\alpha,a} f, \psi \rangle.$$

So, we prove the above asertion.

Definition 3.3 *Let $f \in \mathcal{D}'_{L^{q_1}}$ and $a \in \mathcal{D}'_{L^{q_2}}$, then*

$$H_\alpha(Cd\, f, Cd\, a) = p.v. \int Cd\, f(x-t) Cd\, a(x+\alpha t) \frac{dt}{t}.$$

Theorem 3.3 *Let $f \in \mathcal{D}'_{L^{q_1}}$ and $a \in \mathcal{D}'_{L^{q_2}}$, then $H_\alpha(Cd\, f, Cd\, a)$ is associated with $Cd\, H^*_\alpha(f, A_\varepsilon)$.*

Proof: Let $f \in \mathcal{D}'_{L^{q_1}}$, and $a \in \mathcal{D}'_{L^{q_2}}$. Then, for every $\psi \in \mathcal{D}_{L^{p_1}}$, when $A_\varepsilon = a * \delta_\varepsilon$, using Th.3 (ii) in [3] we get

$$\begin{aligned}\langle (H^*_\alpha(f, A_\varepsilon)) * \delta_\varepsilon, \psi\rangle =& \langle H^*_\alpha(f, A_\varepsilon), \psi * \delta_\varepsilon\rangle \\ =& \langle f, H_\alpha(\psi * \delta_\varepsilon, A_\varepsilon)\rangle.\end{aligned}$$

Then

$$\lim_{\varepsilon\to 0} \langle f, H_\alpha(\psi * \delta_\varepsilon, A_\varepsilon)\rangle = \langle H^*_\alpha(f, A_\varepsilon), \psi\rangle.$$

From the other side we have:

$$\langle H_\alpha(f * \delta_\varepsilon, A_\varepsilon), \psi\rangle = \langle -H^*_{-1-\alpha}(f * \delta_\varepsilon, A_\varepsilon), \psi\rangle$$

$$= \langle f * \delta_\varepsilon, -H_{-1-\alpha}(\psi, A_\varepsilon)\rangle = \langle f, (-H_{-1-\alpha}(\psi, A_\varepsilon)) * \delta_\varepsilon\rangle,$$

and if $\varepsilon \to 0$ then

$$\lim_{\varepsilon\to 0} \langle f, -H_{-1-\alpha}(\psi, A_\varepsilon)\rangle = \langle -H^*_{-1-\alpha}(f, A_\varepsilon), \psi\rangle.$$

So, we prove the above asertion.

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БИЛИНЕАРНА ХИЛБЕРТОВА ТРАНСФОРМАЦИЈА ВО $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$

Анета Бучковска

Р е з и м е

Во овој труд се разгледува билинеарната Хилбертова трансформација во Коломбоовите алгебри $\mathcal{G}_{L^{p_1}} \times \mathcal{G}_{L^{p_2}}$. Релациите меѓу билинеарната Хилбертова трансформација и соодветните дефиниции во Коломбоовите алгебри се дадени преку специјални регуларизации.

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