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ON (r+is)-ANALYTICAL FUNCTIONS WITH LINEAR CHARACTERISTIC FUNCTION

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Abstract. In this work, we determinated generalisated analytical functions in an areolar form of the third class with linear characteristic function.

It is known that the equation

$$\frac{\partial w}{\partial \bar{z}} = Aw + B\bar{w} + F, \tag{1}$$

where

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{2}$$

is the operator derivative in $\bar{z}=x-iy$ of the function $w=u(x,y)+iv(x,y)$ in the simple connected domain D, define different classes of generalised analytical functions in the sense of G.N.Položii [1]. For example, the equation (1):

a) if $A = F \equiv 0$ and $B = \mu = r+is$ is a continuous function of the complex variable $z=x+iy$ in D, determine so called generalised analytical functions of the third class or (r+is)-analytical functions in D with given characteristic $\mu = r+is$;

b) if $F \equiv 0$, $A = A(z)$ and $B = B(z)$ are continuous functions in D determine so called generalised analytic functions of the fourth class;

c) if $A = B \equiv 0$ and $F = F(z)$ is an analytic function in D, then (1) determine so called beanalytic functions (in the sense of Goursatt [2]) e.t.c.

In the sense of itterations, the equation (1) was treated by I.N.Vekua [3].

In this work we established the problem to determine the generalised analytical functions of the third class with linear characteristic function $\mu = r+is = az+b$, which are defined by equation

$$\frac{\partial w}{\partial \bar{z}} = (az+b)\bar{w} \qquad (a,b\in\mathbb{C}) \tag{3}$$

The problem will be solved in two steps.

<u>Step 1</u>. Since in the known books there are not algorithams about the quadratures of the equations

$$F(z,\bar{z},\bar{w},\frac{\partial w}{\partial \bar{z}}) = 0,$$

where the unknown function w is under complex conjugation, we using the method of areolar series [4,5] and we look for solution of the equation

$$\frac{\partial w}{\partial \bar{z}} = \lambda z\bar{w} \qquad (\lambda\in\mathbb{C}) \tag{4}$$

in the form

$$w = \sum_{p,q=0}^{\infty} C_{pq} z^p \bar{z}^q \tag{5}$$

If we put the function (5) into (4) we get the identity

$$\sum_{p=0,q=1}^{\infty} qC_{pq} z^p \bar{z}^{q-1} \equiv \lambda z \sum_{p,q=0}^{\infty} \bar{C}_{pq} \bar{z}^p z^q$$

Since the coefficients of $z^{\alpha}\bar{z}^{\beta}$ in two sides of the identity must be equal, we get the infinite system:

$$\begin{array}{rl}
C_{00} & - \text{ arbitrary} \\
\hline
C_{10} & - \text{ arbitrary} \\
C_{01} & = 0 \\
\hline
C_{20} & - \text{ arbitrary} \\
C_{11} & - \lambda\bar{C}_{00} \\
2C_{02} & = 0 \\
\hline
C_{30} & - \text{ arbitrary} \\
C_{21} & = \lambda\bar{C}_{01} \\
2C_{12} & = \lambda\bar{C}_{10} \\
3C_{03} & = 0 \\
\hline
\vdots & \\
\hline
C_{n0} & - \text{ arbitrary} \\
C_{n-1\ 1} & = \lambda\bar{C}_{0\ n-2} \\
2C_{n-2\ 2} & = \lambda\bar{C}_{1\ n-3} \\
3C_{n-3\ 3} & = \lambda\bar{C}_{2\ n-4} \qquad (n=1,2,\dots) \\
\vdots & \\
(n-1)C_{1\ n-1} & = \lambda\bar{C}_{n-2\ 0} \\
nC_{0n} & = 0 \\
\hline
\vdots &
\end{array} \tag{6}$$

(a) the coefficients

$$C_{po} \qquad (p=0,1,2,\ldots)$$

are arbitrary

(b) the coefficients

$$C_{oq} \qquad (q=1,2,\ldots)$$

are zero, which implies

$$\begin{aligned}
C_{n\ 1} &= 0 \\
C_{2\ n+1} &= 0 \\
C_{n+2\ 3} &= 0 \\
C_{4\ n+3} &= 0 \qquad (n=2,3,\ldots) \\
C_{n+4\ 5} &= 0 \\
C_{6\ n+5} &= 0 \\
&\vdots
\end{aligned}$$

(c) the coefficients C_{pq}, which does not appear in (a) and (b) can be expressed in an uniquelly way by the above coefficients C_{po} by formulas

$$\begin{aligned}
C_{1n} &= \frac{1}{n}\lambda\overline{C}_{n-1\ o} \\
C_{n+1\ 2} &= \frac{1}{2}\lambda\overline{C}_{1\ n} \\
C_{3\ n+2} &= \frac{1}{n+2}\lambda\overline{C}_{n+1\ 2} \\
C_{n+3\ 4} &= \frac{1}{4}\lambda\overline{C}_{3\ n+2} \\
C_{5\ n+4} &= \frac{1}{n+4}\lambda\overline{C}_{n+3\ 4} \\
&\vdots \\
C_{k\ n+k-1} &= \frac{1}{n+k-1}\lambda\overline{C}_{n+k-2\ k-1} \\
C_{n+k\ k+1} &= \frac{1}{k+1}\lambda\overline{C}_{k\ n+k-1} \\
&\vdots
\end{aligned} \tag{7}$$

To state precisely, the formulas (7) by succesive change, give us to express the coefficients of the left side of these equations via arbitrary coefficients $C_{n-1\ o}$. This is valid for n=1,2,...

The areolar series(5) in development form (6) is

$$
\begin{aligned}
w = {} & C_{oo} \\
& + C_{1o} z + 0 \\
& + C_{2o} z^2 + C_{11} z\bar{z} + 0 \\
& + C_{3o} z^3 + 0 + C_{12} z\bar{z}^2 + 0 \\
& + C_{4o} z^4 + 0 + C_{22} z^2\bar{z}^2 + C_{13} z\bar{z}^3 + 0 \\
& + C_{5o} z^5 + 0 + C_{32} z^3\bar{z}^2 + 0 + C_{14} z\bar{z}^4 + 0 \\
& + C_{6o} z^6 + 0 + C_{42} z^4\bar{z}^2 + C_{33} z^3\bar{z}^3 + 0 + C_{15} z\bar{z}^5 + 0 \\
& + C_{7o} z^7 + 0 + C_{52} z^5\bar{z}^2 + 0 + C_{34} z^3\bar{z}^4 + 0 + C_{16} z\bar{z}^6 + 0 \\
& + C_{8o} z^8 + 0 + C_{62} z^6\bar{z}^2 + 0 + C_{44} z^4\bar{z}^4 + C_{35} z^3\bar{z}^5 + 0 + C_{17} z\bar{z}^7 + \\
& + \\
& \vdots
\end{aligned}
\tag{8}
$$

If we classify the members in w in a way like in (8) and after that by formulas (7) express all coefficients C_{pq} via arbitrary $C_{n-1\ o}$ (n=1,2,...) we get

$$
\begin{aligned}
w = {} & \sum_{p=o}^{\infty} C_{po} z^p + \lambda z \sum_{p=o}^{\infty} \bar{C}_{po} \frac{\bar{z}^{p+1}}{p+1} + \\
& + |\lambda|^2 z \frac{\bar{z}^2}{2} \sum_{p=o}^{\infty} C_{po} \frac{z^{p+1}}{p+1} + |\lambda|^2 \lambda \frac{z^3}{3} \sum_{p=o}^{\infty} \bar{C}_{po} \frac{\bar{z}^{p+3}}{(p+1)(p+3)} + \\
& + |\lambda|^4 z \frac{\bar{z}^4}{2\cdot 4} \sum_{p=o}^{\infty} C_{po} \frac{z^{p+3}}{(p+1)(p+3)} + |\lambda|^4 \lambda \frac{z^5}{2\cdot 4} \sum_{p=o}^{\infty} \bar{C}_{po} \frac{\bar{z}^{p+5}}{(p+1)(p+3)(p+5)} + \\
& + |\lambda|^6 z \frac{\bar{z}^6}{2\cdot 4\cdot 6} \sum_{p=o}^{\infty} C_{po} \frac{z^{p+5}}{(p+1)(p+3)(p+5)} + |\lambda|^6 \lambda \frac{z^7}{2\cdot 4\cdot 6} \sum_{p=o}^{\infty} \bar{C}_{po} \frac{\bar{z}^{p+7}}{(p+1)(p+3)(p+5)(p+7)} + \\
& + \\
& \vdots
\end{aligned}
\tag{9}
$$

We take the coefficients C_{po} (p=0,1,2,...) arbitrary, so that the series

$$\sum_{p=o}^{\infty} C_{po} z^p$$

converge in some disc $D=\{z: |z| < R\}$. In the disc D, the series define an analytic function $\phi(z)$:

$$\sum_{p=o}^{\infty} C_{po} z^p = \phi(z) \text{ - arbitrary analytic function}$$

We can integrate this serie in D term by term and so the formula (9) can be write in the following form

$$\begin{aligned} w = {} & \phi(z) + \lambda z \int \overline{\phi}(z) d\overline{z} + \\ & + |\lambda|^2 z \frac{\overline{z}^2}{2} \int \phi(z) dz + |\lambda|^2 \lambda \frac{z^3}{3} \int \overline{z} d\overline{z} \int \overline{\phi}(z) d\overline{z} + \\ & + |\lambda|^4 z \frac{\overline{z}^4}{2\cdot 4} \int z dz \int \phi(z) dz + |\lambda|^4 \lambda \frac{z^5}{2\cdot 4} \int \overline{z} d\overline{z} \int \overline{z} d\overline{z} \int \overline{\phi}(z) d\overline{z} + \\ & + |\lambda|^6 z \frac{\overline{z}^6}{2\cdot 4\cdot 6} \int z dz \int z dz \int \phi(z) dz + |\lambda|^6 \lambda \frac{z^7}{2\cdot 4\cdot 6} \int \overline{z} d\overline{z} \int \overline{z} d\overline{z} \int \overline{z} d\overline{z} \int \overline{\phi}(z) d\overline{z} + \\ & + \\ & \vdots \end{aligned}$$

or, shortly,

$$\begin{aligned} w = {} & \phi(z) + \lambda z \int \overline{\phi}(z) d\overline{z} + \\ & + z \sum_{n=1}^{\infty} \frac{|\lambda|^{2n}}{2^n \cdot n!} [\overline{z}^{2n} \underbrace{\int z dz \int z dz \ldots \int z dz}_{n} \int \phi(z) dz + \\ & + \lambda z^{2n} \underbrace{\int \overline{z} d\overline{z} \int \overline{z} d\overline{z} \ldots \int \overline{z} d\overline{z} \int \overline{z} d\overline{z}}_{n+1} \int \overline{\phi}(z) d\overline{z}] \end{aligned} \tag{10}$$

Since the integral of an analytic function is also an analytic function the succesive integrals in (10) are bounded in every closed domain $D^* = \{z: |z| \le \rho < R\} \subset D$. Since the serie

$$(M+|\lambda|N) \frac{|\lambda|^{2n} \rho^{2n}}{2^n \cdot n!}$$

converge, the estimate

$$\left|\frac{|\lambda|^{2n}}{2^n\cdot n!}[\bar z^{2n}\underbrace{\int zdz\int zdz\ldots\int zdz}_{n}\int\phi(z)dz\right. +$$

$$\left. + \lambda z^{2n}\underbrace{\int \bar z d\bar z\int \bar z d\bar z\ldots\int \bar z d\bar z\int \bar z d\bar z}_{n+1}\int\bar\phi(z)d\bar z]\right| \leq$$

$$\leq (M+|\lambda|N)\frac{|\lambda|^{2n}\rho^{2n}}{2^n.n!}$$

implies that the serie in (10) converge in every closed subject of D.

<u>Theorem 1</u>. Generalised analytic function of the third class with characteristic $\mu = r+is = \lambda z$, defined by equation (4), are given by the formula (10). The function $\phi(z)$ is an arbitrary analytic function.

<u>Step 2</u>. In the equation (3) make the change

$$az + b = \zeta \tag{11}$$

By the operation rulles for the operator derivative $\frac{\partial}{\partial \bar z}$ and (11), we have

$$\frac{\partial w}{\partial \bar z} = \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial \bar z} + \frac{\partial w}{\partial \bar\zeta}\frac{\partial \bar\zeta}{\partial \bar z} = \frac{\partial w}{\partial \zeta}.0 + \frac{\partial w}{\partial \bar\zeta}(\overline{\frac{\partial \zeta}{\partial z}}) = \frac{\partial w}{\partial \bar\zeta}.\bar a.$$

So, the equation (3) is transformated in

$$\frac{\partial w}{\partial \bar\zeta} = \frac{1}{\bar a}\zeta\bar w. \tag{12}$$

Since the equation (12) has the form (4) $(\lambda=\frac{1}{\bar a})$ and is solved it in the step I, we get the following

<u>Theorem 2</u>. The function

$$w = \phi(az+b) + \frac{az+b}{\bar a}[\int\bar\phi(\zeta)d\bar\zeta]_{\zeta=az+b} +$$

$$+ (az+b)\sum_{n=1}^{\infty}\frac{1}{2^n\cdot n!\cdot|a|^{2n}}\{(\overline{az+b})^{2n}[\underbrace{\int\zeta d\zeta\int\zeta d\zeta\ldots\int\zeta d\zeta}_{n}\int\phi(\zeta)d\zeta]_{\zeta=az+b} +$$

$$+ \frac{(az+b)^{2n}}{\bar a}[\underbrace{\int\bar\zeta d\bar\zeta\int\bar\zeta d\bar\zeta\ldots\int\bar\zeta d\bar\zeta\int\bar\zeta d\bar\zeta}_{n+1}\int\bar\phi(\zeta)d\bar\zeta]_{\zeta=az+b}\}, \tag{13}$$

such that ϕ is an arbitrary analytic function of its argument, is generalised analytic function of the third class with characteristic $\mu = r+is = az+b$ linear function from $z = x+iy$.

Here $[\qquad]_{\zeta=az+b}$. signes that after solving these integrals, on the place of ζ we put az+b.

Note 1. By the connection

$$B = 2\frac{\partial}{\partial \bar{z}}$$

which exist between the areolar derivative (2) and Bilimoviches operator B [6] as a measure of disagree of the analicity of nonanalytical functions in the way of the papers of S. Fempl [7,8,9], the functions (10) and (13) can be intepreted also as a class of nonanalytical functions which disagreement of the analicity is proportional to it's complex conjugate value. So then, the coefficient of the proportionality is a linear function of $z = x+iy$, and this value is a constant for the operation $\frac{\partial}{\partial \bar{z}}$.

Note 2. If we put $\lambda = \alpha+i\beta$, $z = x+iy$ and $w = u+iv$, then the equation (4) is a complex form of the partial differention system of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 2(\alpha x-\beta y)u + 2(\beta x+\alpha y)v$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2(\beta x+\alpha y)u - 2(\alpha x-\beta y)v.$$

The solution of this system is

$$u = \mathrm{Re}\,w$$

$$v = \mathrm{Im}\,w,$$

where w is the function defined by (10).

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ЗА (r+is)-АНАЛИТИЧКИ ФУНКЦИИ СО КАРАКТЕРИСТИКА ЛИНЕАРНА ФУНКЦИЈА

Б.Ц. Илиевски

Р е з и м е

Во оваа работа, со метода на ареоларни редови, се определени обопштени аналитички функции од трета класа (10) и (13) со карактеристика μ линеарна функција λz и $az+b$ соодветно.

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