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SOME NEW HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES TO PRODUCTS OF TWO GENERALIZED $(r; g, s, m, \varphi)$-PREINVEX FUNCTIONS

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Abstract. In the present paper, a new class of generalized $(r; g, s, m, \varphi)$-preinvex functions is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving products of two generalized $(r; g, s, m, \varphi)$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities to products of two generalized $(r; g, s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals are established. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities and also extend some results appeared in the literature (see [1]). Some conclusions and future research are also given.

1. Introduction and Preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions (see [2], [3], [15]-[24]).

Fractional calculus (see [16]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. *Here* $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [14],[16]).

Now, let us recall some definitions of various convex functions.

Definition 2. *(see [5]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0,+\infty)$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 3. *(see [6]) A function $f : [0,+\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \geq 0$, $\lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $[0,+\infty)$ as usual. The s-convex functions in the second sense have been investigated in (see [6]).

Definition 4. *(see [7]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details (see [7],[8]).

Definition 5. *(see [9]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star|f|$ (see [10]).
Recently, Liu (see [11]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 2 of P-function.
Also in (see [12]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized $(r; g, s, m, \varphi)$-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving products of two generalized $(r; g, s, m, \varphi)$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type integral inequalities to products of two generalized $(r; g, s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals are given. In Section 4, some conclusions and future research are also given. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula for products of two generalized $(r; g, s, m, \varphi)$-preinvex functions and Hermite-Hadamard type inequalities via Riemann-Liouville fractional integral.

2. New integral inequalities to products of two generalized $(r; g, s, m, \varphi)$-preinvex functions

Definition 6. *(see [4]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1. *In Definition 6, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.*

Definition 7. *(see [13]) A positive function f on the invex set K is said to be logarithmically preinvex, if*

$$f(u + t\eta(v,u)) \leq f^{1-t}(u) f^t(v)$$

for all $u, v \in K$ and $t \in [0,1]$.

Definition 8. *(see [13]) The function f on the invex set K is said to be r-preinvex with respect to η, if*

$$f(u+t\eta(v,u)) \leq M_r(f(u),f(v);t)$$

holds for all $u,v \in K$ and $t \in [0,1]$, where

$$M_r(x,y;t) = \begin{cases} \left[(1-t)x^r + ty^r\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ x^{1-t}y^t, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers x and y.

We next give new definition, to be referred as generalized $(r;g,s,m,\varphi)$-preinvex function.

Definition 9. *Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$, $g : [0,1] \longrightarrow [0,1]$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0,+\infty)$ is said to be generalized $(r;g,s,m,\varphi)$-preinvex with respect to η, if*

$$f\left(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m)\right) \leq M_r(f(\varphi(x)),f(\varphi(y)),m,s;g(t)) \quad (2.1)$$

holds for any fixed $s,m \in (0,1]$ and for all $x,y \in I, t \in [0,1]$, where

$$M_r(f(\varphi(x)),f(\varphi(y)),m,s;g(t))$$

$$= \begin{cases} \left[m(1-g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y))\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \left[f(\varphi(x))\right]^{m(1-g(t))^s}\left[f(\varphi(y))\right]^{g^s(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2. *In Definition 9, it is worthwhile to note that the class of generalized $(r;g,s,m,\varphi)$-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 3. Also, for $r = 1$, $g(t) = t$, $\forall t \in [0,1]$ and $\varphi(x) = x$, $\forall x \in I$, we get the notion of generalized (s,m)-preinvex function (see [4]).*

In this section, in order to prove our main results regarding some new integral inequalities involving products of two generalized $(r;g,s,m,\varphi)$-preinvex functions, we need the following new interesting lemma:

Lemma 1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f,h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ are continuous functions on K° with respect to the same $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Then for any fixed $m \in (0,1]$ and $p,q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1}$$

$$\times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))$$
$$\times h(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)].$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$
$$=\eta(\varphi(b),\varphi(a),m)\int_0^1 (m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m)-m\varphi(a))^p$$
$$\times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-m\varphi(a)-g(t)\eta(\varphi(b),\varphi(a),m))^q$$
$$\times f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))h(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$
$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}$$
$$\times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))$$
$$\times h(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)].$$

□

The following definition will be used in the sequel.

Definition 10. *The Euler beta function is defined for $x, y > 0$ as*

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f, h : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ are continuous functions on K° with $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $k>1, r>1$ and $r^{-1}+l^{-1}=1$. If $f^{\frac{k}{k-1}}, h^{\frac{k}{k-1}}$ are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K° with respect to the same $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$
$$\leq \left(\frac{1}{2}\right)^{\frac{k-1}{k}} \eta(\varphi(b),\varphi(a),m)^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$
$$\times\Bigg[\left(\frac{r}{2s+r}\right)\Bigg\{m\left((1-g(0))^{\frac{2s}{r}+1}-(1-g(1))^{\frac{2s}{r}+1}\right)^{\frac{r}{2}} f^{\frac{rk}{k-1}}(\varphi(a))$$
$$+\left(g^{\frac{2s}{r}+1}(1)-g^{\frac{2s}{r}+1}(0)\right)^{\frac{r}{2}} f^{\frac{rk}{k-1}}(\varphi(b))\Bigg\}^{\frac{2}{r}}$$

$$+\left(\frac{l}{2s+l}\right)\left\{m\left((1-g(0))^{\frac{2s}{l}+1}-(1-g(1))^{\frac{2s}{l}+1}\right)^{\frac{l}{2}}h^{\frac{lk}{k-1}}(\varphi(a))\right.$$

$$\left.\left.+\left(g^{\frac{2s}{l}+1}(1)-g^{\frac{2s}{l}+1}(0)\right)^{\frac{l}{2}}h^{\frac{lk}{k-1}}(\varphi(b))\right\}^{\frac{2}{l}}\right]^{\frac{k-1}{k}}, \tag{2.2}$$

where $B(g(t);k,p,q)=\int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)]$.

Proof. Let $k>1$ and $r>1$. Since $f^{\frac{k}{k-1}}$ and $h^{\frac{k}{k-1}}$ are respectively nonnegative generalized $(r;g,s,m,\varphi)$-preinvex function and nonnegative generalized $(l;g,s,m,\varphi)$-preinvex function on an open m-invex set K°, combining with Lemma 1, Hölder inequality, Cauchy and Minkowski inequality for all $t\in[0,1]$ and for any fixed $s,m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\le|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)]\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))\right.$$

$$\left.\times h^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\le\eta(\varphi(b),\varphi(a),m)^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\int_0^1\left(m(1-g(t))^s f^{\frac{rk}{k-1}}(\varphi(a))+g^s(t)f^{\frac{rk}{k-1}}(\varphi(b))\right)^{\frac{1}{r}}\right.$$

$$\left.\times\left(m(1-g(t))^s h^{\frac{lk}{k-1}}(\varphi(a))+g^s(t)h^{\frac{lk}{k-1}}(\varphi(b))\right)^{\frac{1}{l}}d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\le\left(\frac{1}{2}\right)^{\frac{k-1}{k}}\eta(\varphi(b),\varphi(a),m)^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\int_0^1\left(m(1-g(t))^s f^{\frac{rk}{k-1}}(\varphi(a))+g^s(t)f^{\frac{rk}{k-1}}(\varphi(b))\right)^{\frac{2}{r}}d[g(t)]\right.$$

$$\left.+\int_0^1\left(m(1-g(t))^s h^{\frac{lk}{k-1}}(\varphi(a))+g^s(t)h^{\frac{lk}{k-1}}(\varphi(b))\right)^{\frac{2}{l}}d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\le\left(\frac{1}{2}\right)^{\frac{k-1}{k}}\eta(\varphi(b),\varphi(a),m)^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\Bigg[\Bigg\{\left(\int_0^1 m^{\frac{2}{r}}(1-g(t))^{\frac{2s}{r}} f^{\frac{2k}{k-1}}(\varphi(a))d[g(t)]\right)^{\frac{r}{2}}$$

$$+\left(\int_0^1 g^{\frac{2s}{r}}(t) f^{\frac{2k}{k-1}}(\varphi(b))d[g(t)]\right)^{\frac{r}{2}}\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{\left(\int_0^1 m^{\frac{2}{l}}(1-g(t))^{\frac{2s}{l}} h^{\frac{2k}{k-1}}(\varphi(a))d[g(t)]\right)^{\frac{l}{2}}$$

$$+\left(\int_0^1 g^{\frac{2s}{l}}(t) h^{\frac{2k}{k-1}}(\varphi(b))d[g(t)]\right)^{\frac{l}{2}}\Bigg\}^{\frac{2}{l}}\Bigg]^{\frac{k-1}{k}}$$

$$=\left(\frac{1}{2}\right)^{\frac{k-1}{k}}\eta(\varphi(b),\varphi(a),m)^{p+q+1}B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\Bigg[\left(\frac{r}{2s+r}\right)\Bigg\{m\left((1-g(0))^{\frac{2s}{r}+1}-(1-g(1))^{\frac{2s}{r}+1}\right)^{\frac{r}{2}} f^{\frac{rk}{k-1}}(\varphi(a))$$

$$+\left(g^{\frac{2s}{r}+1}(1)-g^{\frac{2s}{r}+1}(0)\right)^{\frac{r}{2}} f^{\frac{rk}{k-1}}(\varphi(b))\Bigg\}^{\frac{2}{r}}$$

$$+\left(\frac{l}{2s+l}\right)\Bigg\{m\left((1-g(0))^{\frac{2s}{l}+1}-(1-g(1))^{\frac{2s}{l}+1}\right)^{\frac{l}{2}} h^{\frac{lk}{k-1}}(\varphi(a))$$

$$+\left(g^{\frac{2s}{l}+1}(1)-g^{\frac{2s}{l}+1}(0)\right)^{\frac{l}{2}} h^{\frac{lk}{k-1}}(\varphi(b))\Bigg\}^{\frac{2}{l}}\Bigg]^{\frac{k-1}{k}}.$$

□

Corollary 1. *Under the same conditions as in Theorem 2 for $r=l=2$ and $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq\left(\frac{1}{2(s+1)}\right)^{\frac{k-1}{k}}\beta^{\frac{1}{k}}(kp+1,kq+1)|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}$$

$$\times\left[m\left(f^{\frac{2k}{k-1}}(\varphi(a))+h^{\frac{2k}{k-1}}(\varphi(a))\right)+\left(f^{\frac{2k}{k-1}}(\varphi(b))+h^{\frac{2k}{k-1}}(\varphi(b))\right)\right]^{\frac{k-1}{k}}.$$

Theorem 3. *Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a differentiable function. Assume that $f,h: K=[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ are continuous functions on K° with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $l\geq 1$, $r>1$ and $r^{-1}+r_1^{-1}=1$. If f^l,h^l are respectively nonnegative generalized $(r;g,s,m,\varphi)$-preinvex function and nonnegative generalized $(r_1;g,s,m,\varphi)$-preinvex function on an open m-invex* *set K° with respect to the same $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta(\varphi(b),\varphi(a),m)^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q) \times\Bigg[\Bigg\{ m f^{rl}(\varphi(a)) B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2p,2(q+s)\right) + f^{rl}(\varphi(b)) B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(p+s),2q\right)\Bigg\}^{\frac{2}{r}} + \Bigg\{ m h^{r_1 l}(\varphi(a)) B^{\frac{r_1}{2}}\left(g(t);\frac{1}{r_1},2p,2(q+s)\right) + h^{r_1 l}(\varphi(b)) B^{\frac{r_1}{2}}\left(g(t);\frac{1}{r_1},2(p+s),2q\right)\Bigg\}^{\frac{2}{r_1}}\Bigg]^{\frac{1}{l}}. \tag{2.3}$$

Proof. Let $l \geq 1$ and $r > 1$. Since f^l and h^l are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(r_1; g, s, m, \varphi)$-preinvex function on an open m-invex set K°, combining with Lemma 1, the well-known power mean inequality, Cauchy and Minkowski inequality for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)]\right]^{\frac{l-1}{l}} \times\Bigg[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m)) \times h^l(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\Bigg]^{\frac{1}{l}}$$

$$\leq \eta(\varphi(b),\varphi(a),m)^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q) \times\Bigg[\int_0^1 g^p(t)(1-g(t))^q \left(m(1-g(t))^s f^{rl}(\varphi(a)) + g^s(t) f^{rl}(\varphi(b))\right)^{\frac{1}{r}} \times\left(m(1-g(t))^s h^{r_1 l}(\varphi(a)) + g^s(t) h^{r_1 l}(\varphi(b))\right)^{\frac{1}{r_1}} d[g(t)]\Bigg]^{\frac{1}{l}}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{1}{k}}(g(t); 1, p, q)$$

$$\times \left[\int_0^1 \left(m g^p(t)(1-g(t))^{q+s} f^{rl}(\varphi(a)) + g^{p+s}(t)(1-g(t))^q f^{rl}(\varphi(b)) \right)^{\frac{2}{r}} d[g(t)] \right.$$

$$\left. + \int_0^1 \left(m g^p(t)(1-g(t))^{q+s} h^{r_1 l}(\varphi(a)) + g^{p+s}(t)(1-g(t))^q h^{r_1 l}(\varphi(b)) \right)^{\frac{2}{r_1}} d[g(t)] \right]^{\frac{1}{l}}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{l-1}{l}}(g(t); 1, p, q)$$

$$\times \left[\left\{ \left(\int_0^1 m^{\frac{2}{r}} g^{\frac{2p}{r}}(t)(1-g(t))^{\frac{2(q+s)}{r}} f^{2l}(\varphi(a)) d[g(t)] \right)^{\frac{r}{2}} \right. \right.$$

$$\left. + \left(\int_0^1 g^{\frac{2(p+s)}{r}}(t)(1-g(t))^{\frac{2q}{r}} f^{2l}(\varphi(b)) d[g(t)] \right)^{\frac{r}{2}} \right\}^{\frac{2}{r}}$$

$$+ \left\{ \left(\int_0^1 m^{\frac{2}{r_1}} g^{\frac{2p}{r_1}}(t)(1-g(t))^{\frac{2(q+s)}{r_1}} h^{2l}(\varphi(a)) d[g(t)] \right)^{\frac{r_1}{2}} \right.$$

$$\left. \left. + \left(\int_0^1 g^{\frac{2(p+s)}{r_1}}(t)(1-g(t))^{\frac{2q}{r_1}} h^{2l}(\varphi(b)) d[g(t)] \right)^{\frac{r_1}{2}} \right\}^{\frac{2}{r_1}} \right]^{\frac{1}{l}}$$

$$= \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{l-1}{l}}(g(t); 1, p, q)$$

$$\times \left[\left\{ m f^{rl}(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2p, 2(q+s) \right) \right. \right.$$

$$\left. + f^{rl}(\varphi(b)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(p+s), 2q \right) \right\}^{\frac{2}{r}}$$

$$+ \left\{ m h^{r_1 l}(\varphi(a)) B^{\frac{r_1}{2}} \left(g(t); \frac{1}{r_1}, 2p, 2(q+s) \right) \right.$$

$$\left. \left. + h^{r_1 l}(\varphi(b)) B^{\frac{r_1}{2}} \left(g(t); \frac{1}{r_1}, 2(p+s), 2q \right) \right\}^{\frac{2}{r_1}} \right]^{\frac{1}{l}}.$$

□

Corollary 2. *Under the same conditions as in Theorem 3 for $r = r_1 = 2$ and $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta(\varphi(b),\varphi(a),m)^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\Big[m\beta(p+1,q+s+1)\left(f^{2l}(\varphi(a))+h^{2l}(\varphi(a))\right)$$

$$+\beta(p+s+1,q+1)\left(f^{2l}(\varphi(b))+h^{2l}(\varphi(b))\right)\Big]^{\frac{1}{l}}.$$

3. Hermite-Hadamard type fractional integral inequalities to products of two generalized $(r;g,s,m,\varphi)$-preinvex functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities to products of two nonnegative generalized $(r;g,s,m,\varphi)$-preinvex functions via fractional integrals.

Theorem 4. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1]$ with $m\varphi(x) < m\varphi(x)+\eta(\varphi(y),\varphi(x),m)$. Assume that $f,h : K = [m\varphi(x), m\varphi(x)+\eta(\varphi(y),\varphi(x),m)] \longrightarrow (0,+\infty)$ are respectively nonnegative generalized $(r;g,s,m,\varphi)$-preinvex function and nonnegative generalized $(l;g,s,m,\varphi)$-preinvex function on an open m-invex set K°. Then for $\alpha > 0$, $, r > 1$ and $r^{-1}+l^{-1}=1$, we have*

$$\frac{1}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t-m\varphi(x))^{\alpha-1}f(t)h(t)dt$$

$$\leq \frac{1}{2}\Bigg[\Bigg\{mf^r(\varphi(x))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha-1),2s\right)$$

$$+f^r(\varphi(y))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+s-1),0\right)\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{mh^l(\varphi(x))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha-1),2s\right)$$

$$+h^l(\varphi(y))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+s-1),0\right)\Bigg\}^{\frac{2}{l}}\Bigg]. \tag{3.1}$$

Proof. Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since f and h are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K°, combining with Cauchy and Minkowski inequalities for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\frac{1}{\eta^{\alpha}(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt$$

$$= \int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))$$

$$\times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)]$$

$$\leq \int_0^1 g^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)}(t) \Big[m(1 - g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y))\Big]^{\frac{1}{r}}$$

$$\times \Big[m(1 - g(t))^s h^l(\varphi(x)) + g^s(t) h^l(\varphi(y))\Big]^{\frac{1}{l}} d[g(t)]$$

$$\leq \frac{1}{2}\Bigg\{ \int_0^1 \Big[m g^{\alpha-1}(t)(1 - g(t))^s f^r(\varphi(x)) + g^{\alpha+s-1}(t) f^r(\varphi(y))\Big]^{\frac{2}{r}} d[g(t)]$$

$$+ \int_0^1 \Big[m g^{\alpha-1}(t)(1 - g(t))^s h^l(\varphi(x)) + g^{\alpha+s-1}(t) h^l(\varphi(y))\Big]^{\frac{2}{l}} d[g(t)] \Bigg\}$$

$$\leq \frac{1}{2}\Bigg[\Bigg\{ \left(\int_0^1 m^{\frac{2}{r}} g^{\frac{2(\alpha-1)}{r}}(t)(1 - g(t))^{\frac{2s}{r}} f^2(\varphi(x)) d[g(t)]\right)^{\frac{r}{2}}$$

$$+ \left(\int_0^1 g^{\frac{2(\alpha+s-1)}{r}}(t) f^2(\varphi(y)) d[g(t)]\right)^{\frac{r}{2}} \Bigg\}^{\frac{2}{r}}$$

$$+ \Bigg\{ \left(\int_0^1 m^{\frac{2}{l}} g^{\frac{2(\alpha-1)}{l}}(t)(1 - g(t))^{\frac{2s}{l}} h^2(\varphi(x)) d[g(t)]\right)^{\frac{l}{2}}$$

$$+ \left(\int_0^1 g^{\frac{2(\alpha+s-1)}{l}}(t) h^2(\varphi(y)) d[g(t)]\right)^{\frac{l}{2}} \Bigg\}^{\frac{2}{l}} \Bigg]$$

$$= \frac{1}{2}\Bigg[\Bigg\{ m f^r(\varphi(x)) B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2(\alpha - 1), 2s\right)$$

$$+ f^r(\varphi(y)) B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2(\alpha + s - 1), 0\right) \Bigg\}^{\frac{2}{r}}$$

$$+ \Bigg\{ m h^l(\varphi(x)) B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2(\alpha - 1), 2s\right)$$

$$+h^l(\varphi(y))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+s-1),0\right)\Bigg\}^{\frac{2}{l}}\Bigg].$$

□

Corollary 3. *Under the same conditions as in Theorem 4 for $m = s = 1, \varphi(x) = x, g(t) = t$ and $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$, we get (see [1], Theorem 3.3).*

Corollary 4. *Under the same conditions as in Theorem 4 for $r = l = 2$ and $g(t) = t$, we get*

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}J^{\alpha}_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))-}f(m\varphi(x))h(m\varphi(x))$$
$$\leq \frac{1}{2}\Big[m\beta(\alpha,s+1)\left(f^2(\varphi(x))+h^2(\varphi(x))\right)+\beta(\alpha+s,1)\left(f^2(\varphi(y))+h^2(\varphi(y))\right)\Big].$$

Theorem 5. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \longrightarrow (0, +\infty)$ are continuous functions on K° with $m\varphi(x) < m\varphi(x) + \eta(\varphi(y), \varphi(x), m)$. Let $0 < r, l \leq 1, q > 1$ and $p^{-1} + q^{-1} = 1$. If f^p, h^q are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K° with respect to the same $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$, then for $\alpha > 0$, we have*

$$\frac{1}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)}(t-m\varphi(x))^{\alpha-1}f(t)h(t)dt$$
$$\leq\Bigg[mf^{rp}(\varphi(x))B^r\left(g(t);\frac{1}{r},rp(\alpha-1),s\right)$$
$$+f^{rp}(\varphi(y))B^r\left(g(t);\frac{1}{r},s+rp(\alpha-1),0\right)\Bigg]^{\frac{1}{rp}}$$
$$\times\left[mh^{lq}(\varphi(x))B^l\left(g(t);\frac{1}{l},0,s\right)+h^{lq}(\varphi(y))B^l\left(g(t);\frac{1}{l},s,0\right)\right]^{\frac{1}{lq}}. \quad (3.2)$$

Proof. Let $0 < r, l \leq 1, q > 1$ and $p^{-1} + q^{-1} = 1$. Since f^p, h^q are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K°, combining with Hölder and Minkowski inequalities for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\frac{1}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)}(t-m\varphi(x))^{\alpha-1}f(t)h(t)dt$$
$$=\int_0^1 g^{\alpha-1}(t)f(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))$$

$$\times h(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]$$
$$\leq \left(\int_0^1 g^{p(\alpha-1)}(t)f^p(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 h^q(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1 g^{p(\alpha-1)}(t)\Big[m(1-g(t))^s f^{rp}(\varphi(x))+g^s(t)f^{rp}(\varphi(y))\Big]^{\frac{1}{r}}d[g(t)]\right)^{\frac{1}{p}}$$
$$\times \left(\Big[m(1-g(t))^s h^{lq}(\varphi(x))+g^s(t)h^{lq}(\varphi(y))\Big]^{\frac{1}{l}}d[g(t)]\right)^{\frac{1}{q}}$$
$$\leq \Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{p(\alpha-1)}(t)(1-g(t))^{\frac{s}{r}}f^p(\varphi(x))d[g(t)]\right)^r$$
$$+\left(\int_0^1 g^{p(\alpha-1)+\frac{s}{r}}(t)f^p(\varphi(y))d[g(t)]\right)^r\Bigg\}^{\frac{1}{rp}}$$
$$\times\Bigg\{\left(\int_0^1 m^{\frac{1}{l}}(1-g(t))^{\frac{s}{l}}h^q(\varphi(x))d[g(t)]\right)^l$$
$$+\left(\int_0^1 g^{\frac{s}{l}}(t)h^q(\varphi(y))d[g(t)]\right)^l\Bigg\}^{\frac{1}{lq}}$$
$$=\Bigg[mf^{rp}(\varphi(x))B^r\left(g(t);\frac{1}{r},rp(\alpha-1),s\right)$$
$$+f^{rp}(\varphi(y))B^r\left(g(t);\frac{1}{r},s+rp(\alpha-1),0\right)\Bigg]^{\frac{1}{rp}}$$
$$\times\Bigg[mh^{lq}(\varphi(x))B^l\left(g(t);\frac{1}{l},0,s\right)+h^{lq}(\varphi(y))B^l\left(g(t);\frac{1}{l},s,0\right)\Bigg]^{\frac{1}{lq}}.$$

□

Corollary 5. *Under the same conditions as in Theorem 5 for $p=q=2$ and $g(t)=t$, we get*

$$\frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y),\varphi(x),m)}J^\alpha_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))-}f(m\varphi(x))h(m\varphi(x))$$
$$\leq \beta^{\frac{1}{2}}\left(1,\frac{s}{l}+1\right)\Big[mh^{2l}(\varphi(x))+h^{2l}(\varphi(y))\Big]^{\frac{1}{2l}}$$
$$\times\Big[mf^{2r}(\varphi(x))\beta^r\left(2(\alpha-1)+1,\frac{s}{r}+1\right)$$
$$+f^{2r}(\varphi(y))\beta^r\left(\frac{s}{r}+2(\alpha-1)+1,1\right)\Big]^{\frac{1}{2r}}.$$

Theorem 6. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \longrightarrow (0, +\infty)$ are continuous functions on K° with $m\varphi(x) < m\varphi(x) + \eta(\varphi(y), \varphi(x), m)$. Let $q \geq 1, r > 1$ and $r^{-1} + l^{-1} = 1$. If f, h^q are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K° with respect to the same $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$, then for $\alpha > 0$, we have*

$$\frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[mf^r(\varphi(x))B^r\left(g(t); \frac{1}{r}, r(\alpha-1), s\right)\right.$$
$$\left. + f^r(\varphi(y))B^r\left(g(t); \frac{1}{r}, s + r(\alpha-1), 0\right)\right]^{\frac{q-1}{rq}}$$
$$\times \left[\left\{ mf^r(\varphi(x))B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2(\alpha-1), 2s\right)\right.\right.$$
$$\left. + f^r(\varphi(y))B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2(\alpha+s-1), 0\right)\right\}^{\frac{2}{r}}$$
$$+ \left\{ mh^{lq}(\varphi(x))B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2(\alpha-1), 2s\right)\right.$$
$$\left.\left. + h^{lq}(\varphi(y))B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2(\alpha+s-1), 0\right)\right\}^{\frac{2}{l}}\right]^{\frac{1}{q}}. \tag{3.3}$$

Proof. Let $q \geq 1, r > 1$ and $r^{-1} + l^{-1} = 1$. Since f and h^q are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(l; g, s, m, \varphi)$-preinvex function on an open m-invex set K°, combining with the well-known power mean inequality, Cauchy and Minkowski inequalities for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt$$
$$= \int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))$$
$$\times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)]$$
$$\leq \left(\int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$
\begin{aligned}
&\times\Bigg[\int_0^1 g^{\alpha-1}(t)f(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))\\
&\quad\times h^q(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\Bigg]^{\frac{1}{q}}\\
&\le\left(\int_0^1 g^{\alpha-1}(t)\Big[m(1-g(t))^s f^r(\varphi(x))+g^s(t)f^r(\varphi(y))\Big]^{\frac{1}{r}}d[g(t)]\right)^{1-\frac{1}{q}}\\
&\times\Bigg\{\int_0^1 g^{\alpha-1}(t)\Big[m(1-g(t))^s f^r(\varphi(x))+g^s(t)f^r(\varphi(y))\Big]^{\frac{1}{r}}\\
&\quad\times\Big[m(1-g(t))^s h^{lq}(\varphi(x))+g^s(t)h^{lq}(\varphi(y))\Big]^{\frac{1}{l}}d[g(t)]\Bigg\}^{\frac{1}{q}}\\
&\le\left(\frac{1}{2}\right)^{\frac{1}{q}}\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{\alpha-1}(t)(1-g(t))^{\frac{s}{r}}f(\varphi(x))d[g(t)]\right)^r\\
&\quad+\left(\int_0^1 g^{\alpha-1+\frac{s}{r}}(t)f(\varphi(y))d[g(t)]\right)^r\Bigg\}^{\frac{q-1}{rq}}\\
&\times\Bigg\{\int_0^1\Big[mg^{\alpha-1}(t)(1-g(t))^s f^r(\varphi(x))+g^{\alpha+s-1}(t)f^r(\varphi(y))\Big]^{\frac{2}{r}}d[g(t)]\\
&+\int_0^1\Big[mg^{\alpha-1}(t)(1-g(t))^s h^{lq}(\varphi(x))+g^{\alpha+s-1}(t)h^{lq}(\varphi(y))\Big]^{\frac{2}{l}}d[g(t)]\Bigg\}^{\frac{1}{q}}\\
&\le\left(\frac{1}{2}\right)^{\frac{1}{q}}\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{\alpha-1}(t)(1-g(t))^{\frac{s}{r}}f(\varphi(x))d[g(t)]\right)^r\\
&\quad+\left(\int_0^1 g^{\alpha-1+\frac{s}{r}}(t)f(\varphi(y))d[g(t)]\right)^r\Bigg\}^{\frac{q-1}{rq}}\\
&\times\Bigg[\Bigg\{\left(\int_0^1 m^{\frac{2}{r}}g^{\frac{2(\alpha-1)}{r}}(t)(1-g(t))^{\frac{2s}{r}}f^2(\varphi(x))d[g(t)]\right)^{\frac{r}{2}}\\
&\quad+\left(\int_0^1 g^{\frac{2(\alpha+s-1)}{r}}(t)f^2(\varphi(y))d[g(t)]\right)^{\frac{r}{2}}\Bigg\}^{\frac{2}{r}}\\
&+\Bigg\{\left(\int_0^1 m^{\frac{2}{l}}g^{\frac{2(\alpha-1)}{l}}(t)(1-g(t))^{\frac{2s}{l}}h^{2q}(\varphi(x))d[g(t)]\right)^{\frac{l}{2}}\\
&\quad+\left(\int_0^1 g^{\frac{2(\alpha+s-1)}{l}}(t)h^{2q}(\varphi(y))d[g(t)]\right)^{\frac{l}{2}}\Bigg\}^{\frac{2}{l}}\Bigg]^{\frac{1}{q}}
\end{aligned}
$$

$$= \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[m f^r(\varphi(x)) B^r \left(g(t); \frac{1}{r}, r(\alpha-1), s \right) \right.$$

$$\left. + f^r(\varphi(y)) B^r \left(g(t); \frac{1}{r}, s + r(\alpha-1), 0 \right) \right]^{\frac{q-1}{rq}}$$

$$\times \left[\left\{ m f^r(\varphi(x)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha-1), 2s \right) \right. \right.$$

$$\left. + f^r(\varphi(y)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha+s-1), 0 \right) \right\}^{\frac{2}{r}}$$

$$+ \left\{ m h^{lq}(\varphi(x)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha-1), 2s \right) \right.$$

$$\left. \left. + h^{lq}(\varphi(y)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha+s-1), 0 \right) \right\}^{\frac{2}{l}} \right]^{\frac{1}{q}}.$$

□

Corollary 6. *Under the same conditions as in Theorem 6 for $m = q = s = 1, \varphi(x) = x, g(t) = t$ and $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$, we get (see* [1], *Theorem 3.9). Also for $q = 1$, we get Theorem 4.*

Corollary 7. *Under the same conditions as in Theorem 6 for $r = l = 2$ and $g(t) = t$, we get*

$$\frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y), \varphi(x), m)} J^\alpha_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))-} f(m\varphi(x)) h(m\varphi(x))$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \left[m f^2(\varphi(x)) \beta^2 \left(\alpha, \frac{s}{2} + 1 \right) + f^2(\varphi(y)) \beta^2 \left(\alpha + \frac{s}{2}, 1 \right) \right]^{\frac{q-1}{2q}}$$

$$\times \left[m\beta(\alpha, s+1) \left(f^2(\varphi(x)) + h^{2q}(\varphi(x)) \right) \right.$$

$$\left. + \beta(\alpha+s, 1) \left(f^2(\varphi(y)) + h^{2q}(\varphi(y)) \right) \right]^{\frac{1}{q}}.$$

Remark 3. *For $\alpha > 0$, for different choices of positive values $r, l = \frac{1}{2}, \frac{1}{3}, 2$, etc., for any fixed $s, m \in (0, 1]$, for a particular choices of a differentiable function $g(t) = e^{-t}, \ln(t+1), \sin\left(\frac{\pi t}{2}\right), \cos\left(\frac{\pi t}{2}\right)$, etc, and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, x^n for all $x > 0$ and for all $n \in \mathbb{N}$, etc, by Theorem 4, Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities to products of two nonnegative generalized $(r; g, s, m, \varphi)$-preinvex functions.*

4. Conclusions

In this paper, we proved some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving products of two generalized $(r; g, s, m, \varphi)$-preinvex functions. Also, we established some new Hermite-Hadamard type integral inequalities to products of two generalized $(r; g, s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals. These results not only extend the results appeared in the literature (see [1]), but also provide new estimates on these types.

Motivated by this new interesting class of generalized $(r; g, s, m, \varphi)$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard and Ostrowski type integral inequalities to products of various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, q-calculus, (p, q)-calculus, time scale calculus and conformable fractional integrals.

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