Abstract. In the present paper, a new class of generalized \((r; g, s, m, \varphi)\)-preinvex functions is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving products of two generalized \((r; g, s, m, \varphi)\)-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities to products of two generalized \((r; g, s, m, \varphi)\)-preinvex functions via Riemann-Liouville fractional integrals are established. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities and also extend some results appeared in the literature (see [1]). Some conclusions and future research are also given.

1. Introduction and Preliminaries

The following notations are used throughout this paper. We use \(I\) to denote an interval on the real line \(\mathbb{R} = (-\infty, +\infty)\) and \(I^o\) to denote the interior of \(I\). For any subset \(K \subseteq \mathbb{R}^n\), \(K^o\) is used to denote the interior of \(K\). \(\mathbb{R}^n\) is used to denote a \(n\)-dimensional vector space. The set of integrable functions on the interval \([a, b]\) is denoted by \(L_1[a, b]\).

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.** Let \(f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}\) be a convex function on an interval \(I\) of real numbers and \(a, b \in I\) with \(a < b\). Then the following inequality holds:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2}.
\]

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In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions (see [2], [3], [15]-[24]).

Fractional calculus (see [16]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

**Definition 1.** Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J^\alpha_{a+}f \) and \( J^\alpha_{b-}f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a
\]

and

\[
J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,
\]

where \( \Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du \). Here \( J^0_{a+}f(x) = J^0_{b-}f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [14],[16]).

Now, let us recall some definitions of various convex functions.

**Definition 2.** (see [5]) A nonnegative function \( f : I \subseteq \mathbb{R} \rightarrow [0, +\infty) \) is said to be P-function or P-convex, if

\[
f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \; t \in [0, 1].
\]

**Definition 3.** (see [6]) A function \( f : [0, +\infty) \rightarrow \mathbb{R} \) is said to be \( s \)-convex in the second sense, if

\[
f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)
\]

for all \( x, y \geq 0, \; \lambda \in [0, 1] \) and \( s \in (0, 1] \).

It is clear that a 1-convex function must be convex on \([0, +\infty)\) as usual. The \( s \)-convex functions in the second sense have been investigated in (see [6]).

**Definition 4.** (see [7]) A set \( K \subseteq \mathbb{R}^n \) is said to be invex with respect to the mapping \( \eta : K \times K \rightarrow \mathbb{R}^n \), if \( x + t\eta(y, x) \in K \) for every \( x, y \in K \) and \( t \in [0, 1] \).

Notice that every convex set is invex with respect to the mapping \( \eta(y, x) = y - x \), but the converse is not necessarily true. For more details (see [7],[8]).
Definition 5. (see [9]) The function $f$ defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to $\eta$, if for every $x, y \in K$ and $t \in [0, 1]$, we have that
$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following
$$\int_{a}^{b} (x - a)^p(b - x)^q f(x)dx = \sum_{k=0}^{\infty} B_{m,k} f(\gamma_k) + R_m^*|f|, \quad (1.3)$$
for certain $B_{m,k}, \gamma_k$ and rest $R_m^*|f|$ (see [10]).

Recently, Liu (see [11]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 2 of $P$-function. Also in (see [12]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized $(r; g, s, m, \varphi)$-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving products of two generalized $(r; g, s, m, \varphi)$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type integral inequalities to products of two generalized $(r; g, s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals are given. In Section 4, some conclusions and future research are also given. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula for products of two generalized $(r; g, s, m, \varphi)$-preinvex functions and Hermite-Hadamard type inequalities via Riemann-Liouville fractional integral.

2. NEW INTEGRAL INEQUALITIES TO PRODUCTS OF TWO GENERALIZED $(r; g, s, m, \varphi)$-PREINVEX FUNCTIONS

Definition 6. (see [4]) A set $K \subseteq \mathbb{R}^n$ is said to be $m$-invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 1. In Definition 6, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the $m$-invex set degenerates an invex set on $K$.

Definition 7. (see [13]) A positive function $f$ on the invex set $K$ is said to be logarithmically preinvex, if
$$f(u + t\eta(v, u)) \leq f^{1-t}(u)f^t(v)$$
for all $u, v \in K$ and $t \in [0, 1]$. 
Definition 8. (see [13]) The function $f$ on the invex set $K$ is said to be $r$-preinvex with respect to $\eta$, if

$$f(u + t\eta(v, u)) \leq M_r(f(u), f(v); t)$$

holds for all $u, v \in K$ and $t \in [0, 1]$, where

$$M_r(x, y; t) = \left\{ \begin{array}{ll}
(1 - t)x^r + ty^r, & \text{if } r \neq 0; \\
|x^{1-t}y^t|, & \text{if } r = 0,
\end{array} \right.$$ 

is the weighted power mean of order $r$ for positive numbers $x$ and $y$.

We next give new definition, to be referred as generalized $(r; g, s, m, \varphi)$-preinvex function.

Definition 9. Let $K \subseteq \mathbb{R}^n$ be an open $m$-invex set with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$, $g : [0, 1] \rightarrow (0, 1]$ be a differentiable function and $\varphi : I \rightarrow K$ is a continuous function. The function $f : K \rightarrow (0, +\infty)$ is said to be generalized $(r; g, s, m, \varphi)$-preinvex with respect to $\eta$, if

$$f \left( m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m) \right) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, s; g(t))$$

holds for any fixed $s, m \in (0, 1]$ and for all $x, y \in I, t \in [0, 1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m, s; g(t)) = \left\{ \begin{array}{ll}
[m(1 - g(t))s f^r(\varphi(x)) + g^s(t)f^r(\varphi(y))]^{\frac{1}{r}}, & \text{if } r \neq 0; \\
[f(\varphi(x))]^{m(1 - g(t))s}[f(\varphi(y))]^{g^s(t)}, & \text{if } r = 0,
\end{array} \right.$$

is the weighted power mean of order $r$ for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2. In Definition 9, it is worthwhile to note that the class of generalized $(r; g, s, m, \varphi)$-preinvex function is a generalization of the class of $s$-convex in the second sense function given in Definition 3. Also, for $r = 1$, $g(t) = t$, $\forall t \in [0, 1]$ and $\varphi(x) = x$, $\forall x \in I$, we get the notion of generalized $(s, m)$-preinvex function (see [4]).

In this section, in order to prove our main results regarding some new integral inequalities involving products of two generalized $(r; g, s, m, \varphi)$-preinvex functions, we need the following new interesting lemma:

Lemma 1. Let $\varphi : I \rightarrow K$ be a continuous function and $g : [0, 1] \rightarrow [0, 1]$ is a differentiable function. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow \mathbb{R}$ are continuous functions on $K^c$ with respect to the same $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Then for any fixed $m \in (0, 1]$ and $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx = \eta(\varphi(b), \varphi(a), m)^p q + 1.$$
\[ \times \int_{0}^{1} g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \]
\[ \times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)]. \]

**Proof.** It is easy to observe that
\[ \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} \]
\[ (x - m\varphi(a))^p(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx \]
\[ = \eta(\varphi(b), \varphi(a), m) \int_{0}^{1} (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \]
\[ \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q \]
\[ \times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)] \]
\[ = \eta(\varphi(b), \varphi(a), m)^{p+q+1} \]
\[ \times \int_{0}^{1} g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \]
\[ \times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)]. \]

\[ \square \]

The following definition will be used in the sequel.

**Definition 10.** The Euler beta function is defined for \( x, y > 0 \) as
\[ \beta(x, y) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \]

**Theorem 2.** Let \( \varphi : I \rightarrow K \) be a continuous function and \( g : [0, 1] \rightarrow [0, 1] \) is a differentiable function. Assume that \( f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow (0, +\infty) \) are continuous functions on \( K^o \) with \( m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m) \). Let \( k > 1, r > 1 \) and \( r^{-1} + l^{-1} = 1 \). If \( f, h \) are respectively nonnegative generalized \( (r; g, s, m, \varphi) \)-preinvex function and nonnegative generalized \( (l; g, s, m, \varphi) \)-preinvex function on an open \( m \)-invex set \( K^o \) with respect to the same \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \) for any fixed \( s, m \in (0, 1] \), then for any fixed \( p, q > 0 \), we have
\[ \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} \]
\[ (x - m\varphi(a))^p(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx \]
\[ \leq \left( \frac{1}{2} \right)^{\frac{k-1}{2}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B_\frac{1}{k}(g(t); k, p, q) \]
\[ \times \left[ \left( \frac{r}{2s + r} \right) \left\{ m \left( 1 - g(0) \right)^{\frac{p+1}{k}} - \left( 1 - g(1) \right)^{\frac{p+1}{k}} \right\} \right] \frac{1}{k}f_{\frac{1}{k}}(\varphi(a)) \]
\[ + \left( g^{\frac{p+1}{r}}(1 - g^{\frac{p+1}{r}}(0)) \right)^{\frac{1}{r}} f_{\frac{1}{r}}(\varphi(b)) \]
\[
\begin{align*}
&+ \left( \frac{l}{2s+1} \right) \left\{ m \left( (1 - g(0))^{\frac{q}{r}+1} - (1 - g(1))^{\frac{q}{r}+1} \right) \right. \\
&\quad + \left( g^{\frac{q}{r}+1}(1) - g^{\frac{q}{r}+1}(0) \right) \left. \right\} \frac{1}{p} h^{\frac{1}{p}-r}((\varphi(a)) \\
&\quad + \left( g^{\frac{q}{r}+1}(1) - g^{\frac{q}{r}+1}(0) \right) \left. \right\} \frac{1}{p} h^{\frac{1}{p}-r}((\varphi(b)) \right) \frac{1}{k-1},
\end{align*}
\]

where \( B(g(t); k, p, q) = \int_0^1 g^{kp}(t)(1 - g(t))^{kq}d[g(t)] \).

Proof. Let \( k > 1 \) and \( r > 1 \). Since \( f^{\frac{1}{r+1}} \) and \( h^{\frac{1}{r+1}} \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^o \), combining with Lemma 1, H"older inequality, Cauchy and Minkowski inequality for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1] \), we get

\[
\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p(m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx
\]

\[
\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[ \int_0^1 g^{kp}(t)(1 - g(t))^{kq}d[g(t)] \right] \frac{1}{p}
\]

\[
\times \left[ \int_0^1 f^{\frac{1}{r+1}}(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))
\right. \\
\left. \times h^{\frac{1}{r+1}}(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)] \right] \frac{1}{k-1}
\]

\[
\leq \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^\frac{1}{k}(g(t); k, p, q)
\]

\[
\times \left[ \int_0^1 \left( m(1 - g(t))^s f^{\frac{1}{r+1}}(\varphi(a)) + g^s(t) f^{\frac{1}{r+1}}(\varphi(b)) \right)^\frac{1}{r} d[g(t)] \right]
\]

\[
\times \left( m(1 - g(t))^s h^{\frac{1}{r+1}}(\varphi(a)) + g^s(t) h^{\frac{1}{r+1}}(\varphi(b)) \right)^\frac{1}{k-1} d[g(t)]
\]

\[
\leq \left( \frac{1}{2} \right)^{\frac{k-1}{k}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^\frac{1}{k}(g(t); k, p, q)
\]

\[
\times \left[ \int_0^1 \left( m(1 - g(t))^s f^{\frac{1}{r+1}}(\varphi(a)) + g^s(t) f^{\frac{1}{r+1}}(\varphi(b)) \right)^\frac{1}{r} d[g(t)] \right]
\]

\[
+ \int_0^1 \left( m(1 - g(t))^s h^{\frac{1}{r+1}}(\varphi(a)) + g^s(t) h^{\frac{1}{r+1}}(\varphi(b)) \right)^\frac{1}{k-1} d[g(t)]
\]

\[
\leq \left( \frac{1}{2} \right)^{\frac{k-1}{k}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^\frac{1}{k}(g(t); k, p, q)
\]
\[
\times \left\{ \left( \int_0^1 m^\frac{s}{r}(1 - g(t))^{\frac{2s}{r}} f^\frac{2s}{k-1}(\varphi(a))d[g(t)] \right)^{\frac{r}{k}} \right. \\
+ \left. \left( \int_0^1 g^\frac{2s}{r}(t) f^\frac{2s}{k-1}(\varphi(b))d[g(t)] \right)^{\frac{r}{k}} \right\}^{\frac{k}{k-1}} \\
+ \left\{ \left( \int_0^1 m^\frac{s}{r}(1 - g(t))^{\frac{2s}{r}} h^\frac{2s}{k-1}(\varphi(a))d[g(t)] \right)^{\frac{r}{k}} \right. \\
+ \left. \left( \int_0^1 g^\frac{2s}{r}(t) h^\frac{2s}{k-1}(\varphi(b))d[g(t)] \right)^{\frac{r}{k}} \right\}^{\frac{k}{k-1}} \\
= \left( \frac{1}{2} \right)^{\frac{k-1}{k}} \eta(\varphi(b), \varphi(a), m)^{p+q+1} D^\frac{r}{k}(g(t); k, p, q) \\
\times \left[ \left( \frac{r}{2s} \right) \left\{ m \left( (1 - g(0))^{\frac{2s}{r}} - (1 - g(1))^{\frac{2s}{r}} \right)^{\frac{r}{k}} f^\frac{2s}{k-1}(\varphi(a)) \right. \\
+ \left. \left( g^{\frac{2s}{r}}(1) - g^{\frac{2s}{r}}(0) \right)^{\frac{r}{k}} f^\frac{2s}{k-1}(\varphi(b)) \right\}^{\frac{k}{k-1}} \\
+ \left( \frac{l}{2s+t} \right) \left\{ m \left( (1 - g(0))^{\frac{2s}{r}} - (1 - g(1))^{\frac{2s}{r}} \right)^{\frac{r}{k}} h^\frac{2s}{k-1}(\varphi(a)) \right. \\
+ \left. \left( g^{\frac{2s}{r}}(1) - g^{\frac{2s}{r}}(0) \right)^{\frac{r}{k}} h^\frac{2s}{k-1}(\varphi(b)) \right\}^{\frac{k}{k-1}} \right].
\]
set $K^o$ with respect to the same $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$, then for any fixed $p, q > 0$, we have
\[
\int_{m \varphi(a)}^{m \varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m \varphi(a))^p (m \varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) h(x) dx
\]
\[
\leq \left( \frac{1}{2} \right)^{\frac{1}{r}} \eta(\varphi(b), \varphi(a), m)^p g(t) B^{\frac{1}{r}}(g(t); 1, p, q)
\]
\[
\times \left\{ m f^{r^l}(\varphi(a)) B^{\frac{1}{r}} \left( g(t); \frac{1}{r}, 2p, 2(q + s) \right) \right. 
\]
\[
\left. + f^{r^l}(\varphi(b)) B^{\frac{1}{r}} \left( g(t); \frac{1}{r}, 2p, 2q \right) \right\}^{\frac{1}{r}} 
\]
\[
+ \left\{ m h^{r^l}(\varphi(a)) B^{\frac{1}{r}} \left( g(t); \frac{1}{r_1}, 2p, 2(q + s) \right) \right. 
\]
\[
\left. + h^{r^l}(\varphi(b)) B^{\frac{1}{r}} \left( g(t); \frac{1}{r_1}, 2p, 2q \right) \right\}^{\frac{1}{r_1}} \\cdots \quad (2.3)
\]

Proof. Let $l \geq 1$ and $r > 1$. Since $f^l$ and $h^l$ are respectively nonnegative generalized $(r; g, s, m, \varphi)$-preinvex function and nonnegative generalized $(r_1; g, s, m, \varphi)$-preinvex function on an open $m$-invex set $K^o$, combining with Lemma 1, the well-known power mean inequality, Cauchy and Minkowski inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get
\[
\int_{m \varphi(a)}^{m \varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m \varphi(a))^p (m \varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) h(x) dx
\]
\[
\leq |\eta(\varphi(b), \varphi(a), m)|^{p + q + 1} \left[ \int_0^1 g^p(t) (1 - g(t))^q d[g(t)] \right]^{\frac{1}{r + 1}} 
\]
\[
\times \left[ \int_0^1 g^p(t) (1 - g(t))^q f^{l^l}(m \varphi(a) + g(t) \eta(\varphi(b), \varphi(a), m)) 
\right.
\]
\[
\times h^l(\varphi(a) + g(t) \eta(\varphi(b), \varphi(a), m)) d[g(t)] \right]^{\frac{1}{r}} 
\]
\[
\leq \eta(\varphi(b), \varphi(a), m)^{p + q + 1} B^{\frac{1}{r}}(g(t); 1, p, q)
\]
\[
\times \left[ \int_0^1 g^p(t) (1 - g(t))^q \left( m(1 - g(t))^s f^{l^l}(\varphi(a)) + g^s(t) f^{l^l}(\varphi(b)) \right) \right.
\]
\[
\times \left( m(1 - g(t))^s h^{l^l}(\varphi(a)) + g^s(t) h^{l^l}(\varphi(b)) \right) \left[ \frac{1}{r_1} d[g(t)] \right]^{\frac{1}{r_1}} 
\]
\[
\leq \left( \frac{1}{2} \right)^\frac{t}{\tau} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{1}{\tau}}(g(t); 1, p, q) \\
\times \left[ \int_0^1 \left( m g^p(t)(1 - g(t))^{q+s} f^{\tau l}(\varphi(a)) + g^{p+s}(t)(1 - g(t))^q f^{\tau l}(\varphi(b)) \right)^\frac{2r}{\tau} d[g(t)] \\
+ \int_0^1 \left( m g^p(t)(1 - g(t))^{q+s} h^{\tau l}(\varphi(a)) + g^{p+s}(t)(1 - g(t))^q h^{\tau l}(\varphi(b)) \right)^\frac{2r}{\tau} d[g(t)] \right]^\frac{t}{\tau}
\]

\[
\leq \left( \frac{1}{2} \right)^\frac{t}{\tau} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{1}{\tau}}(g(t); 1, p, q) \\
\times \left[ \left\{ \int_0^1 m^\frac{2r}{\tau} g^{r} g^p(t)(1 - g(t))^{\frac{2(q+s)}{\tau}} f^{2l}(\varphi(a)) d[g(t)] \right\}^\frac{2r}{\tau} \\
+ \left\{ \int_0^1 m^\frac{2r}{\tau} g^{r} g^p(t)(1 - g(t))^{\frac{2(q+s)}{\tau}} h^{2l}(\varphi(a)) d[g(t)] \right\}^\frac{2r}{\tau} \\
+ \left\{ \int_0^1 g^{r} g^{p}(1 - g(t))^{\frac{2(q+s)}{r_1}} h^{2l}(\varphi(b)) d[g(t)] \right\}^\frac{2r}{r_1} \right]^\frac{t}{\tau}
\]

\[
= \left( \frac{1}{2} \right)^\frac{t}{\tau} \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{1}{\tau}}(g(t); 1, p, q) \\
\times \left[ \left\{ m f^{\tau l}(\varphi(a)) B^{\frac{r}{\tau}} \left( g(t); \frac{1}{r}, 2p, 2(q+s) \right) \\
+ f^{\tau l}(\varphi(b)) B^{\frac{r}{\tau}} \left( g(t); \frac{1}{r}, 2(p+s), 2q \right) \right\}^\frac{2r}{\tau} \\
+ \left\{ m h^{\tau l}(\varphi(a)) B^{\frac{r_1}{r}} \left( g(t); \frac{1}{r_1}, 2p, 2(q+s) \right) \\
+ h^{\tau l}(\varphi(b)) B^{\frac{r_1}{r}} \left( g(t); \frac{1}{r_1}, 2(p+s), 2q \right) \right\}^\frac{2r_1}{r_1} \right]^\frac{t}{r_1}.
\]

\[\square\]
Corollary 2. Under the same conditions as in Theorem 3 for \( r = r_1 = 2 \) and \( g(t) = t \), we get

\[
\int_{m\varphi(a)}^{m\varphi(a)+n(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+n(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx
\]

\[
\leq \left( \frac{1}{2} \right)^\frac{q}{2} n(\varphi(b),\varphi(a),m)^p+q+1 \beta_{m\varphi}^{r_{m\varphi}} \left( p+1, q+1 \right)
\times \left[ m\beta(p+1,q+s+1) \left( f^{2l}(\varphi(a)) + h^{2l}(\varphi(a)) \right) \right.
\left. + \beta(p+s+1,q+1) \left( f^{2l}(\varphi(b)) + h^{2l}(\varphi(b)) \right) \right]^{\frac{1}{r}}.
\]

3. Hermite-Hadamard Type Fractional Integral Inequalities to Products of Two Generalized \((r; g, s, m, \varphi)\)-Preinvex Functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities to products of two nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex functions via fractional integrals.

Theorem 4. Let \( \varphi : I \rightarrow K \) be a continuous function and \( g : [0, 1] \rightarrow [0, 1] \) is a differentiable function. Suppose \( K \subseteq \mathbb{R} \) be an open \( m\)-invex subset with respect to \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \) for any fixed \( s, m \in (0, 1) \) with \( m\varphi(x) < m\varphi(x) + \eta(\varphi(y), \varphi(x), m) \). Assume that \( f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \rightarrow (0, +\infty) \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^\circ \). Then for \( \alpha > 0, r > 1 \) and \( r^{-1}+l^{-1}=1 \), we have

\[
\frac{1}{\eta^{\alpha}(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)}^{m\varphi(x)+\eta(\varphi(y), \varphi(x), m)} (t-m\varphi(x))^{\alpha-1} f(t)h(t)dt
\]

\[
\leq \frac{1}{2} \left\{ m f^\alpha(\varphi(x)) B^\frac{r}{2} \left( g(t); \frac{1}{r}, 2(\alpha-1), 2s \right) \right.
+ f^\alpha(\varphi(y)) B^\frac{r}{2} \left( g(t); \frac{1}{r}, 2(\alpha+s-1), 0 \right) \right\]^{\frac{2}{r}}
\]

\[
+ \left\{ m h^\alpha(\varphi(x)) B^\frac{1}{2} \left( g(t); \frac{1}{l}, 2(\alpha-1), 2s \right) \right.
+ h^\alpha(\varphi(y)) B^\frac{1}{2} \left( g(t); \frac{1}{l}, 2(\alpha+s-1), 0 \right) \right\]^{\frac{2}{l}}. \quad (3.1)
\]
Proof. Let \( r > 1 \) and \( r^{-1} + l^{-1} = 1 \). Since \( f \) and \( h \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^\circ \), combining with Cauchy and Minkowski inequalities for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1] \), we get

\[
\frac{1}{\eta^s(\varphi(y), \varphi(x), m)} \int_{m(\varphi(x) + g(0)\eta(\varphi(y), \varphi(x), m))}^{m(\varphi(x) + g(1)\eta(\varphi(y), \varphi(x), m))} (t - m\varphi(x))^{a-1} f(t)h(t) dt
\]

\[
= \int_0^1 g^{a-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))
\times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)]
\leq \int_0^1 g^{(a-1)(\frac{1}{r} + \frac{1}{l})(t)} \left[ m(1 - g(t))^s f^r(\varphi(x)) + g^s(t) f^r(\varphi(y)) \right]^{\frac{l}{r}}
\times \left[ m(1 - g(t))^s h^l(\varphi(x)) + g^s(t) h^l(\varphi(y)) \right]^{\frac{1}{l}} d[g(t)]
\leq \frac{1}{2} \left\{ \left( \int_0^1 m^l \left( 1 - g(t) \right)^{\frac{2(a-1)}{r}} f^2(\varphi(x)) d[g(t)] \right)^{\frac{1}{r}}
+ \left( \int_0^1 g^{\frac{2(a-1)}{r}}(t) f^2(\varphi(y)) d[g(t)] \right)^{\frac{1}{r}} \right\}
+ \left\{ \left( \int_0^1 m^l \left( 1 - g(t) \right)^{\frac{2(a-1)}{r}} h^2(\varphi(x)) d[g(t)] \right)^{\frac{1}{r}}
+ \left( \int_0^1 g^{\frac{2(a-1)}{r}}(t) h^2(\varphi(y)) d[g(t)] \right)^{\frac{1}{r}} \right\}
= \frac{1}{2} \left\{ m f^r(\varphi(x)) B^\frac{r}{2} \left( g(t); \frac{1}{r}, 2(\alpha - 1), 2s \right)
+ f^r(\varphi(y)) B^\frac{r}{2} \left( g(t); \frac{1}{r}, 2(\alpha + s - 1), 0 \right) \right\}^{\frac{2}{r}}
+ \left\{ m h^l(\varphi(x)) B^\frac{l}{2} \left( g(t); \frac{1}{l}, 2(\alpha - 1), 2s \right)
+ h^l(\varphi(y)) B^\frac{l}{2} \left( g(t); \frac{1}{l}, 2(\alpha + s - 1), 0 \right) \right\}^{\frac{1}{l}}
\[ +h'(\varphi(y))B^\frac{1}{l}\left( g(t); \frac{1}{l}, 2(\alpha + s - 1), 0 \right) \] \]

\[ \square \]

**Corollary 3.** Under the same conditions as in Theorem 4 for \( m = s = 1, \varphi(x) = x, g(t) = t \) and \( \eta(\varphi(b), \varphi(a), m) = \eta(b, a) \), we get (see [1], Theorem 3.3).

**Corollary 4.** Under the same conditions as in Theorem 4 for \( r = l = 2 \) and \( g(t) = t \), we get

\[ \frac{\Gamma(\alpha)}{\eta^a(\varphi(y), \varphi(x), m)} \int_{m^\varphi(x) + \eta(\varphi(y), \varphi(x), m)}^1 \left( t - m^\varphi(x) \right)^{\alpha - 1} f(t)h(t)dt \]

\[ \leq \frac{1}{2}\left[ m\beta(\alpha, s + 1) \left( f^2(\varphi(x)) + h^2(\varphi(x)) \right) + \beta(\alpha, s + 1) \left( f^2(\varphi(y)) + h^2(\varphi(y)) \right) \right]. \]

**Theorem 5.** Let \( \varphi : I \rightarrow K \) be a continuous function and \( g : [0, 1] \rightarrow [0, 1] \) is a differentiable function. Assume that \( f, h : K = [m^\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \rightarrow (0, +\infty) \) are continuous functions on \( K^0 \) with \( m^\varphi(x) < m\varphi(x) + \eta(\varphi(y), \varphi(x), m) \). Let \( 0 < r, l \leq 1, q > 1 \) and \( p^{-1} + q^{-1} = 1 \). If \( f^p, h^q \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^0 \) with respect to the same \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \) for any fixed \( s, m \in (0, 1) \), then for \( \alpha > 0 \), we have

\[ \frac{1}{\eta^a(\varphi(y), \varphi(x), m)} \int_{m^\varphi(x) + \eta(\varphi(y), \varphi(x), m)}^1 \left( t - m^\varphi(x) \right)^{\alpha - 1} f(t)h(t)dt \]

\[ \leq \left[ m f^{rp}(\varphi(x)) B^r \left( g(t); \frac{1}{r}, rp(\alpha - 1), s \right) \right]^\frac{1}{rp} \]

\[ +\left[ m h^q(\varphi(y)) B^q \left( g(t); \frac{1}{l}, s + rp(\alpha - 1), 0 \right) \right]^\frac{1}{q}. \]

\[ (3.2) \]

**Proof.** Let \( 0 < r, l \leq 1, q > 1 \) and \( p^{-1} + q^{-1} = 1 \). Since \( f^p, h^q \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^0 \), combining with Hölder and Minkowski inequalities for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1) \), we get

\[ \frac{1}{\eta^a(\varphi(y), \varphi(x), m)} \int_{m^\varphi(x) + \eta(\varphi(y), \varphi(x), m)}^1 \left( t - m^\varphi(x) \right)^{\alpha - 1} f(t)h(t)dt \]

\[ = \int_0^1 g^{\alpha - 1}(t) \left( f(m^\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \right) \]
\[ \times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)] \leq \left( \int_0^1 g^{p(\alpha-1)}(t) f^p(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)] \right)^{\frac{1}{p}} \]

\[ \times \left( \int_0^1 h^q(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)] \right)^{\frac{1}{q}} \]

\[ \leq \left( \int_0^1 g^{p(\alpha-1)}(t) \left[ m(1 - g(t))^s f^{rp}(\varphi(x)) + g^s(t) f^{rp}(\varphi(y)) \right]^\frac{1}{r} d[g(t)] \right)^{\frac{1}{r}} \]

\[ \times \left\{ \left( \int_0^1 m^{\frac{1}{r}}(1 - g(t))^{\frac{1}{r}} h^q(\varphi(x))d[g(t)] \right)^{\frac{1}{l}} + \left( \int_0^1 g^{s}(t) h^q(\varphi(y))d[g(t)] \right)^{\frac{1}{l}} \right\} \]

\[ = \left[ m f^{rp}(\varphi(x)) B^r \left( g(t); \frac{1}{r}, rp(\alpha - 1), s \right) \right. \]

\[ \left. + f^{rp}(\varphi(y)) B^r \left( g(t); \frac{1}{r}, s + rp(\alpha - 1), 0 \right) \right]^{\frac{1}{r}} \]

\[ \times \left[ mh^q(\varphi(x)) B^l \left( g(t); \frac{1}{l}, 0, s \right) + h^q(\varphi(y)) B^l \left( g(t); \frac{1}{l}, s, 0 \right) \right]^{\frac{1}{l}}. \]

**Corollary 5.** Under the same conditions as in Theorem 5 for \( p = q = 2 \) and \( g(t) = t \), we get

\[ \frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y), \varphi(x), m)} J^{\alpha}_{(m\varphi(x) + \eta(\varphi(y), \varphi(x), m))} f(m\varphi(x)) h(m\varphi(x)) \]

\[ \leq \beta^{\frac{1}{r}} \left( 1, \frac{s}{r} + 1 \right) \left[ mh^{2l}(\varphi(x)) + h^{2l}(\varphi(y)) \right]^{\frac{1}{r}} \]

\[ \times \left[ mf^{2r}(\varphi(x)) \beta^r \left( 2(\alpha - 1) + 1, \frac{s}{r} + 1 \right) \right. \]

\[ \left. + f^{2r}(\varphi(y)) \beta^r \left( \frac{s}{r} + 2(\alpha - 1) + 1, 1 \right) \right]^{\frac{1}{r}}. \]
Theorem 6. Let \( \varphi : I \rightarrow K \) be a continuous function and \( g : [0, 1] \rightarrow [0, 1] \) is a differentiable function. Assume that \( f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \rightarrow (0, +\infty) \) are continuous functions on \( K^\circ \) with \( m\varphi(x) + \eta(\varphi(y), \varphi(x), m) \). Let \( q \geq 1, r > 1 \) and \( r^{-1} + l^{-1} = 1 \). If \( f, h^q \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^\circ \) with respect to the same \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R} \) for any fixed \( s, m \in (0, 1] \), then for \( \alpha > 0 \), we have

\[
\frac{1}{\eta^a(\varphi(y), \varphi(x), m)} \int_{m\varphi(x) + g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x) + g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{a-1} f(t) h(t) dt \\
\leq \left( \frac{1}{2} \right)^\frac{1}{q} \left[ m f^r(\varphi(x)) B^r \left( g(t); \frac{1}{r}, r(\alpha - 1), s \right) \\
+ f^r(\varphi(y)) B^r \left( g(t); \frac{1}{r}, s + r(\alpha - 1), 0 \right) \right]^{\frac{1}{q}} \\
\times \left\{ m f^r(\varphi(x)) B^r \left( g(t); \frac{1}{r}, 2(\alpha - 1), 2s \right) \\
+ f^r(\varphi(y)) B^r \left( g(t); \frac{1}{r}, 2(\alpha + s - 1), 0 \right) \right\} \left( \frac{2}{q} \right)^\frac{1}{q}. \tag{3.3}
\]

Proof. Let \( q \geq 1, r > 1 \) and \( r^{-1} + l^{-1} = 1 \). Since \( f \) and \( h^q \) are respectively nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex function and nonnegative generalized \((l; g, s, m, \varphi)\)-preinvex function on an open \( m\)-invex set \( K^\circ \), combining with the well-known power mean inequality, Cauchy and Minkowski inequalities for all \( t \in [0, 1] \) and for any fixed \( s, m \in (0, 1] \), we get

\[
\frac{1}{\eta^a(\varphi(y), \varphi(x), m)} \int_{m\varphi(x) + g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x) + g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{a-1} f(t) h(t) dt \\
= \int_0^1 g^{a-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \\
\times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) dt \\
\leq \left( \int_0^1 g^{a-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) dt \right)^{1 - \frac{1}{q}}.
\]
\[
\begin{align*}
&\times \left[ \int_0^1 g^{\alpha-1}(t)f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))
\times h^q(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)] \right]^\frac{1}{q} \\
&\leq \left( \int_0^1 g^{\alpha-1}(t) \left[ m(1-g(t))^s f^r(\varphi(x)) + g^s(t)f^r(\varphi(y)) \right] \frac{1}{r} d[g(t)] \right)^{1-\frac{1}{q}} \\
&\times \left\{ \int_0^1 g^{\alpha-1}(t) \left[ m(1-g(t))^s f^r(\varphi(x)) + g^s(t)f^r(\varphi(y)) \right] \frac{1}{r} d[g(t)] \right\} \frac{1}{1} \\
&\leq \left( \frac{1}{2} \right)^\frac{1}{q} \left\{ \left( \int_0^1 \frac{1}{r} g^{\alpha-1}(t)(1-g(t))^s f(\varphi(x))d[g(t)] \right)^{r} \\
&+ \left( \int_0^1 g^{\alpha-1}(t)f(\varphi(y))d[g(t)] \right)^{\frac{1}{r}} \right\} \frac{1}{1} \\
&\times \left\{ \left( \int_0^1 \frac{1}{r} g^{\alpha-1}(t)(1-g(t))^s f(\varphi(x))d[g(t)] \right)^{\frac{1}{r}} \\
&+ \left( \int_0^1 g^{\alpha-1}(t)f(\varphi(y))d[g(t)] \right)^{\frac{1}{r}} \right\} \frac{1}{1} \\
&\times \left\{ \left( \int_0^1 \frac{1}{r} g^{\alpha-1}(t)(1-g(t))^s f(\varphi(x))d[g(t)] \right)^{\frac{1}{r}} \\
&+ \left( \int_0^1 g^{\alpha-1}(t)f^2(\varphi(y))d[g(t)] \right)^{\frac{1}{r}} \right\} \frac{1}{1} \\
&\times \left\{ \left( \int_0^1 \frac{1}{r} g^{\alpha-1}(t)(1-g(t))^s f^2(\varphi(x))d[g(t)] \right)^{\frac{1}{r}} \\
&+ \left( \int_0^1 g^{\alpha-1}(t)f^2(\varphi(y))d[g(t)] \right)^{\frac{1}{r}} \right\} \frac{1}{1} \\
&\times \left\{ \left( \int_0^1 \frac{1}{r} g^{\alpha-1}(t)(1-g(t))^s f^2(\varphi(x))d[g(t)] \right)^{\frac{1}{r}} \\
&+ \left( \int_0^1 g^{\alpha-1}(t)f^2(\varphi(y))d[g(t)] \right)^{\frac{1}{r}} \right\} \frac{1}{1} \right\} \frac{1}{1} 
\end{align*}
\]
\[
= \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ m f^\alpha (\varphi(x)) B^\alpha \left( g(t); \frac{1}{r}, r(\alpha - 1), s \right) \\
+ f^\alpha (\varphi(y)) B^\alpha \left( g(t); \frac{1}{r}, s + r(\alpha - 1), 0 \right) \right]^\frac{2^{\frac{1}{q}}}{q}
\times \left\{ m f^\alpha (\varphi(x)) B^\alpha \left( g(t); \frac{1}{r}, 2(\alpha - 1), 2s \right) \\
+ f^\alpha (\varphi(y)) B^\alpha \left( g(t); \frac{1}{r}, 2(\alpha + s - 1), 0 \right) \right\}^{\frac{2}{q}}
\times \left\{ m h^\alpha (\varphi(x)) B^\alpha \left( g(t); \frac{1}{l}, 2(\alpha - 1), 2s \right) \\
+ h^\alpha (\varphi(y)) B^\alpha \left( g(t); \frac{1}{l}, 2(\alpha + s - 1), 0 \right) \right\}^{\frac{2}{q}}.
\]

**Corollary 6.** Under the same conditions as in Theorem 6 for \( m = q = r = l = 1 \), \( \varphi(x) = x, g(t) = t \) and \( \eta(\varphi(b), \varphi(a), m) = \eta(b, a) \), we get (see [1], Theorem 3.9). Also for \( q = 1 \), we get Theorem 4.

**Corollary 7.** Under the same conditions as in Theorem 6 for \( r = l = 2 \) and \( g(t) = t \), we get

\[
\Gamma(\alpha) \eta^\alpha(\varphi(y), \varphi(x), m) \int_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))}^\alpha f(m\varphi(x)) h(m\varphi(x))
\leq \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ m f^2(\varphi(x)) \beta^2 \left( \alpha, \frac{s}{2} + 1 \right) + f^2(\varphi(y)) \beta^2 \left( \alpha + \frac{s}{2}, 1 \right) \right]^{\frac{4-q}{2q}}
\times \left[ m \beta (\alpha, s + 1) \left( f^2(\varphi(x)) + h^2(\varphi(x)) \right) \\
+ \beta (\alpha + s, 1) \left( f^2(\varphi(y)) + h^2(\varphi(y)) \right) \right]^{\frac{1}{q}}.
\]

**Remark 3.** For \( \alpha > 0 \), for different choices of positive values \( r, l = \frac{1}{2}, \frac{1}{3}, 2 \), etc., for any fixed \( s, m \in (0, 1] \), for a particular choices of a differentiable function \( g(t) = e^{-t}, \ln(t+1), \sin \left( \frac{\pi}{2} \right), \cos \left( \frac{\pi}{2} \right) \), etc, and a particular choices of a continuous function \( \varphi(x) = e^x \) for all \( x \in \mathbb{R} \), \( x^n \) for all \( x > 0 \) and for all \( n \in \mathbb{N} \), etc, by Theorem 4, Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities to products of two nonnegative generalized \((r; g, s, m, \varphi)\)-preinvex functions.
4. Conclusions

In this paper, we proved some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving products of two generalized \((r; g, s, m, \varphi)\)-preinvex functions. Also, we established some new Hermite-Hadamard type integral inequalities to products of two generalized \((r; g, s, m, \varphi)\)-preinvex functions via Riemann-Liouville fractional integrals. These results not only extend the results appeared in the literature (see [1]), but also provide new estimates on these types.

Motivated by this new interesting class of generalized \((r; g, s, m, \varphi)\)-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard and Ostrowski type integral inequalities to products of various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, \(k\)-fractional integrals, local fractional integrals, fractional integral operators, \(q\)-calculus, \((p,q)\)-calculus, time scale calculus and conformable fractional integrals.

References


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