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ABOUT THE FORM OF THE SOLUTION OF A THIRD-DEGREE EQUATION WITH FUNCTIONAL COEFFICIENTS

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Abstract. In this work we discuss the following linear differential equation of third degree

$$f(x)y''' + g(x)y'' + h(x)y' + r(x)y = 0,$$

and we transform it under certain circumstances in its canonical form (1) whereupon

$$a(x) = c,$$

which is a differential equation with constant coefficients, and according to the sign of the figuration equations we determine its solution with the formulas (2).

1. Introduction

The following canonical equation of third degree [3]

$$y''' + a(x)y = 0, \tag{1}$$

has three linear independent solutions which are determinated with the formulas:

$$y_1(x) = 1 + \sum_{n=1}^{\infty}(-1)^n \int_o^x\int_o^x\int_o^x a(x)dx^3 \cdots \int_0^x\int_0^x\int_0^x a(x)dx^3,$$

$$y_2(x) = x + \sum_{n=1}^{\infty}(-1)^n \int_0^x\int_0^x\int_0^x a(x)dx^3 \cdots \int_0^x\int_0^x\int_0^x xa(x)dx^3,$$

$$y_3(x) = \frac{x^2}{2!} + \sum_{n=1}^{\infty}(-1)^n \int_0^x\int_0^x\int_0^x a(x)dx^3 \cdots \int_0^x\int_0^x\int_0^x \frac{x^2}{2!}a(x)dx^3.$$

According to the sign of the function they are marked as:

when $a(x) > 0,\ y_1(x) = \cos_{a(x)}^{III} x,\ y_2(x) = \sin_{a(x)}^{III} x,\ y_3(x) = tr_{a(x)}^{III} x.$

when $a(x) < 0,\ y_1(x) = ch_{a(x)}^{III} x,\ y_2(x) = sh_{a(x)}^{III} x,\ y_3(x) = trh_{a(x)}^{III} x.$

Now the solution can be noted down as:

when $a(x) > 0,\ y(x) = C_1 \cos_{a(x)}^{III} x + C_2 \sin_{a(x)}^{III} x + C_3 tr_{a(x)}^{III} x,$

when $a(x) < 0,\ y(x) = C_1 ch_{a(x)}^{III} x + C_2 sh_{a(x)}^{III} x + C_3 trh_{a(x)}^{III} x.$ (2)

For the equation

$$y''' + y = 0 \tag{3}$$

the adequate solutions are:

$$\begin{aligned} y_1(x) &= \cos_1^{III} x = \frac{1}{3}e^{-x} + \frac{2}{3}e^{\frac{x}{2}} \cos\frac{\sqrt{3}}{2}x, \\ y_2(x) &= \sin_1^{III} x = -\frac{1}{3}e^{-x} + \frac{2}{3}e^{\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}), \\ y_3(x) &= tr_1^{III} x = \frac{1}{3}e^{-x} + \frac{2}{3}e^{\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x - \frac{\pi}{6}). \end{aligned} \tag{4}$$

According to (2) the solution can be writen down with the following form:

$$y(x) = C_1 \cos_1^{III} x + C_2 \sin_1^{III} x + C_3 tr_1^{III} x$$

And for the equation

$$y''' - y = 0 \tag{5}$$

the adequate solutions are:

$$\begin{aligned} y_1(x) &= ch_1^{III} x = \frac{1}{3}e^{x} + \frac{2}{3}e^{-\frac{x}{2}} \cos\frac{\sqrt{3}}{2}x, \\ y_2(x) &= sh_1^{III} x = \frac{1}{3}e^{x} - \frac{2}{3}e^{-\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x - \frac{\pi}{6}), \\ y_3(x) &= trh_1^{III} x = \frac{1}{3}e^{x} - \frac{2}{3}e^{-\frac{x}{2}} \sin(\frac{\sqrt{3}}{2}x + \frac{\pi}{6}). \end{aligned} \tag{6}$$

Aften that, according to (2) the solution can be written as:

$$y(x) = C_1 ch_1^{III} x + C_2 sh_1^{III} x + C_3 trh_1^{III} x$$

2. For the differential equation:

$$y''' + c^3 y = 0 \tag{7}$$

according to the previous discussion the adequate solutions are:

a) when $c > 0$

$$\begin{aligned} y_1(x) &= \cos_c^{III} x = \frac{1}{3}e^{-cx} + \frac{2}{3}e^{\frac{cx}{2}} \cos\frac{c\sqrt{3}}{2}x, \\ y_2(x) &= \sin_c^{III} x = -\frac{1}{3}e^{-cx} + \frac{2}{3}e^{\frac{cx}{2}} \sin(\frac{c\sqrt{3}}{2}x + \frac{\pi}{6}), \\ y_3(x) &= tr_c^{III} x = \frac{1}{3}e^{-cx} + \frac{2}{3}e^{\frac{cx}{2}} \sin(\frac{c\sqrt{3}}{2}x - \frac{\pi}{6}). \end{aligned} \tag{7a}$$

The general solution according to (2) will be:

$$y(x) = C_1 \cos_c^{III} x + C_2 \sin_c^{III} x + C_3 tr_c^{III} x. \tag{7b}$$

a) when $c < 0$

$$y_1(x) = ch_c^{III} x = \frac{1}{3}e^{-cx} + \frac{2}{3}e^{\frac{cx}{2}} \cos\frac{c\sqrt{3}}{2}x,$$

$$y_2(x) = sh_c^{III}x = -\frac{1}{3}e^{-cx} - \frac{2}{3}e^{\frac{cx}{2}}\sin(\frac{c\sqrt{3}}{2}x - \frac{\pi}{6}), \tag{7c}$$
$$y_3(x) = trh_c^{III}x = \frac{1}{3}e^{-cx} - \frac{2}{3}e^{\frac{cx}{2}}\sin(\frac{c\sqrt{3}}{2}x + \frac{\pi}{6}).$$

In this case the general solution according to (2) will be:

$$y(x) = C_1ch_c^{III}x + C_2sh_c^{III}x + C_3trh_c^{III}x. \tag{7d}$$

My result. In the differential equation

$$f(x)y''' + g(x)y'' + h(x)y' + r(x)y = 0, \tag{8}$$

We change the variable with in order to transforme it in its canonical form. This change leads us to the equation:

$$y''' + \frac{1}{x'^2}\left[\frac{g(x)}{f(x)}x'^3 - 3x'x''\right]y'' + \frac{1}{x'^2}\left[3x''^2 - x'x''' - \frac{g(x)}{f(x)}x'^2x'' + \frac{h(x)}{f(x)}x'^4\right]y' +$$
$$+ \frac{r(x)}{f(x)}x'^3y = 0$$

This last equation in order to take the form (1) shoud satisfy the following relations:

$$\frac{g(x)}{f(x)}x'^3 - 3x'x'' = 0,$$
$$3x''^2 - x'x''' - \frac{g(x)}{f(x)}x'^2x'' + \frac{h(x)}{f(x)}x'^4 = 0, \qquad (9),$$
$$\frac{r(x)}{f(x)}x'^3 = a(x).$$

From the first and the second equation from the relation (9) we have the condition

$$3\left[\frac{g(x)}{f(x)}\right]' + 2\left[\frac{g(x)}{f(x)}\right]^2 - 9\frac{h(x)}{f(x)} = 0, \tag{10}$$

for deduction of the equation (8) in its canonical form.

From the first and the third relation for the function $a(x)$ we have

$$a(x) = e^{\int\frac{r'f - rf' + gr}{rf}dx}. \tag{11}$$

And for the new independent variable we have:

$$t = \int\left[\sqrt[3]{\frac{r}{f}}e^{-\frac{1}{3}\int\frac{r'f - rf' + gr}{rf}dx}\right]dx. \tag{12}$$

In order to be the function $a(x) = c$, from the first and the third equation of the relations (9) we get the condition

$$r'f - rf' + gr = 0, \tag{13}$$

In this case for the new variable we have the relation:

$$ct = \int\sqrt[3]{\frac{r}{f}}dx. \tag{14}$$

Example. The differential equation

$$x^3y''' + 3x^2y'' + xy' + k^3y = 0, \; k = const. \tag{15}$$

satisfies the conditions (10) and (13), so with the change

$$ct = k \ln x, \tag{16}$$

It transformes in the following differential equation

$$y''' + c^3 y = 0,$$

with the new variable t . We should make a note that c and k have the same sign.

Concedering the formulas (7a) do (7d) and the change (16), the independent variable value for the fundamental solutions and the general solution of the differential equation (15) we get:

a) when $k > 0$ for the fundamental solutions we have:

$$y_1(x) = \cos^{III}_{k \ln x} x = \frac{1}{3}x^{-k} + \frac{2}{3}x^{\frac{c}{2}} \cos \frac{k\sqrt{3}}{2} \ln x,$$

$$y_2(x) = \sin^{III}_{k \ln x} x = -\frac{1}{3}x^{-k} + \frac{2}{3}x^{\frac{c}{2}} \sin(\frac{k\sqrt{3}}{2} \ln x + \frac{\pi}{6}),$$

$$y_3(x) = tr^{III}_{k \ln x} x = \frac{1}{3}x^{-k} + \frac{2}{3}x^{\frac{c}{2}} \sin(\frac{k\sqrt{3}}{2} \ln x - \frac{\pi}{6}).$$

And for the general solution we have:

$$y(x) = C_1 \cos^{III}_{k \ln x} x + C_2 \sin^{III}_{k \ln x} x + C_3 tr^{III}_{k \ln x} x.$$

a) Now, when , for the fundamental solutions we have:

$$y_1(x) = ch^{III}_{k \ln x} x = \frac{1}{3}x^{-k} + \frac{2}{3}x^{\frac{c}{2}} \cos \frac{k\sqrt{3}}{2} \ln x,$$

$$y_2(x) = sh^{III}_{k \ln x} x = -\frac{1}{3}x^{-k} + \frac{2}{3}x^{\frac{c}{2}} \sin(\frac{k\sqrt{3}}{2} \ln x - \frac{\pi}{6}),$$

$$y_3(x) = trh^{III}_{k \ln x} x = \frac{1}{3}x^{-k} - \frac{2}{3}x^{\frac{c}{2}} \sin(\frac{k\sqrt{3}}{2} \ln x + \frac{\pi}{6}).$$

And for the general solution we have:

$$y(x) = C_1 ch^{III}_{k \ln x} x + C_2 sh^{III}_{k \ln x} x + C_3 trh^{III}_{k \ln x} x.$$

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