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A CLASS OF NON-STANDARD SPECTRAL PROBLEMS

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Abstract

We consider parabolic equations in two dimensions with interface corresponding to concentrated heat capacity and singular own source. We study non-standard spectral problems in which the eigenvalue appears in the conjugation conditions or at the boundary of the spatial domain.

Let us consider the parabolic initial boundary value problem for the heat equation with concentrated capacity at the interior point $x = \xi$:

$$\begin{aligned} &[1 + K\delta(x - \xi)]\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad Q = (0,1) \times (0,T), \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1}$$

where $K > 0$ and $\delta(x)$ is the Dirac distribution. Similar problems are already mentioned in [2], [3]. The derivations in (1) are taken in the sense of a theory of distributions. It follows from (1), that the solution of this problem satisfies at $(x,t) \in (0,\xi) \times (0,T)$ and $(x,t) \in (\xi,1) \times (0,T)$ the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in (0,\xi) \cup (\xi,1), \quad t \in (0,T)$$

$$u(0,t) = u(1,t) = 0, \;\; u(x,0) = u_0(x),$$

and at $x = \xi$ the conditions of conjugation

$$[u]_\xi \equiv u(\xi+0,t) - u(\xi-0,t) = 0, \quad K\frac{\partial u}{\partial t}(\xi,t) = \left[\frac{\partial u}{\partial x}\right]_\xi . \tag{2}$$

Let us consider for a moment a standard heat equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in Q, \\ &u(0,t) = u(1,t) = 0, \; u(x,0) = u_0(x). \end{aligned} \tag{3}$$

It is well known that, using the Fourier method, the solution of the equation (3) can be presented in the following form

$$u(x,t) \doteq \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}[b_n + v_n(t)] \sin n\pi x, \tag{4}$$

where

$$b_n = 2\int_0^1 u_0(\xi) \sin n\pi\xi \, d\xi,$$

$$v_n(t) = 2\int_0^t e^{n^2\pi^2\tau} \left[\int_0^1 f(\xi,t) \sin\, n\pi\xi \, d\xi\right] d\tau.$$

Setting $u(x,t) = U(x)V(t)$, the following spectral problem can be obtained [1]:

$$-\frac{d^2U(x)}{dx^2} = \lambda U(x), \quad x \in (0,1); \quad U(0) = U(1) = 0. \tag{5}$$

The solutions of this problem are non-trivial if $\lambda > 0$ and the eigenvalues are

$$\lambda = \lambda_n = n^2\pi^2, \qquad n = 1,2,\ldots,$$

while the corresponding eigenfunctions are

$$U = U_n(x) = \sin n\pi x.$$

These functions satisfy the conditions of orthogonality:

$$\int_0^1 U_n(x)U_m(x)\,dx = \begin{cases} \frac{1}{2}, & n = m \\ 0, & n \neq m \end{cases} = \delta_{nm}.$$

We consider an abstract Cauchy problem

$$\frac{du}{dt} + Au = f(t), \quad t \in (0,T), \quad u(0) = u_0, \tag{6}$$

where u, $f : (0,T) \to H$, H is a e Hilbert space, A is a linear selfadjoint unbounded positive definite linear operator with domain $D(A)$ dense in H. The product $(u,v)_A = (Au,v)$, $(u,v \in D(A))$ satisfies the inner product axioms. Reinforcing $D(A)$ in the norm $\|u\|_A = (u,u)^{1/2}$, we obtain a Hilbert space $H_A \subset H$. Operator A extends to mapping $A : H_A \to H_{A^{-1}}$, where $H_{A^{-1}}$ is the adjoint space for H_A, and $H_A \subset H \subset H_{A^{-1}}$ form Geljfand triple.

It is easy to see that the initial boundary value problem (3) can be reduced in form (6) letting

$$A = -\frac{\partial^2}{\partial x^2}, \quad H = L_2(0,1), \quad H_A = \overset{o}{W}{}_2^1\,(0,1).$$

Setting $u = \alpha(t)U$, $U \in H$ and $f = 0$, we obtain

$$u' + Au = 0,$$

i.e.

$$\frac{\alpha'(t)}{\alpha(t)}U + AU = 0.$$

Since $\dfrac{\alpha'(t)}{\alpha(t)} = -\lambda = const. \in R$, the abstract spectral problem is obtained:

$$AU = \lambda U, \quad U \in H. \tag{7}$$

The spectra of (7) is discrete, all eigenvalues $\lambda = \lambda_n, \ n = 1,2,\ldots$ are positive,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_n \to \infty,$$

while the eigenfunctions $U = U_n, \ n = 1,2,\ldots$ satisfy the conditions of orthogonality:

$$(U_n, U_m) = \delta_{nm},$$

and represent the basis of the spaces H and H_A. Now, the solution of the problem (7) can be obtained in the following form

$$u(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \left[c_n + \int_0^t e^{\lambda_n \tau} f_n(\tau) d\tau \right] U_n, \tag{8}$$

where

$$c_n = (u_0, U_n), \quad f_n(t) = (f(t), U_n).$$

Setting $f(t) = 0$ in (6), an a priori estimate can be done. We take inner product in H of (6) with u and apply an inequality $\lambda_1(x,x) \leq (Ax,x)$ to obtain:

$$\left(\frac{du}{dt}, u \right) + (A(t)u, u) = 0,$$

i.e.

$$\frac{1}{2} \frac{d}{dt} [(u,u)] = -(A(t)u,u) = 0 \leq -\lambda_1(u,u).$$

Thus,

$$\frac{d}{dt} \|u\|^2 + 2\lambda_1 \|u\|^2 \leq 0,$$

$$\|u\|^2 \leq C e^{-\int 2\lambda_1 dt} = C e^{-2\lambda_1 t}.$$

From the initial condition $u(0) = u_0$, we get $\|u_0\|^2 = C$, which implies the estimate

$$\|u(t)\| \leq e^{-\lambda_1 t} \|u_0\|. \tag{9}$$

Now, let B be a linear selfadjoint positive definite operator with domain $D(B)$ dense in H, A is a linear selfadjoint unbounded positive definite linear operator with domain $D(A)$ dense in H_B and $A \geq B$. We consider an abstract Cauchy problem:

$$B \frac{du}{dt} + Au = f(t), \quad t \in (0,T); \quad u(0) = u_0. \tag{10}$$

It is easy to see that the problem (10) can be written in the following form

$$B^{1/2}B^{1/2}\frac{du}{dt} + AB^{-1/2}B^{1/2}u = f(t).$$

Setting $\tilde{u} = B^{1/2}u$, $B^{-1/2}AB^{-1/2} = \tilde{A}$ and $B^{-1/2}f(t) = \tilde{f}$, we obtain an abstract Cauchy problem

$$\frac{d\tilde{u}}{dt} + \tilde{A}\tilde{u} = \tilde{f}, \quad t \in (0,T); \quad \tilde{u}(0) = \tilde{u}_0.$$

The spectral problem can be written in the form

$$\tilde{A}\tilde{U} = \tilde{\lambda}\tilde{U}.$$

Therefore, the spectrum is discrete, all eigenvalues $\tilde{\lambda} = \tilde{\lambda}_n, \quad n = 1,2,\ldots$ are positive, $0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots, \tilde{\lambda}_n \to \infty$, while the eigenfunctions $U = U_n, \quad n = 1,2,\ldots$ satisfy the condition of orthogonality $(\tilde{U}_j, \tilde{U}_k) = \delta_{jk}$ and represent the basis of the space H. If we set

$$\tilde{U} = B^{1/2}U, \quad \tilde{U}_n = B^{1/2}U_n$$

the spectral problem takes the following form:

$$AU = \tilde{\lambda}BU. \tag{11}$$

The spectrum is discrete, all eigenvalues are positive, they can be ordered as an increasing sequence, while eigenfunctions $U = U_n, \quad n = 1,2,\ldots$ satisfy the condition of orthogonality

$$(U_j, U_k)_B = (BU_j, U_k) = (\tilde{U}_j, \tilde{U}_k) = \delta_{jk}$$

and represent the basis of the space H_B.

The solution of problem (10) can be written in the form

$$u(t) = \sum_{n=1}^{\infty} e^{-\tilde{\lambda}_n t}\left[c_n + \int_0^t e^{\tilde{\lambda}_n \tau} f_n(\tau)d\tau\right] U_n,$$

where

$$c_n = (u_0, U_n)_B, \quad f_n(t) = (f(t), U_n).$$

In order to obtain an energy estimate for the solution of the problem, when $f(t) = 0$, we take inner product in H_B of (10) with $2u$:

$$2\left(B\frac{du}{dt}, u\right) + 2(Au, u) = 0$$

$$\left(B\frac{du}{dt}, u\right) + \left(Bu, \frac{du}{dt}\right) + 2\|u\|_A^2 = 0$$

$$\frac{d}{dt}[(Bu, u)] + 2\|u\|_A^2 = 0\,.$$

Taking into account an inequality $\|u\|_A^2 \geq \tilde{\lambda}_1\|u\|_B^2$, we get

$$0 = \frac{d}{dt}\|u\|_B^2 + 2\|u\|_A^2 \geq \frac{d}{dt}\|u\|_B^2 + 2\tilde{\lambda}_1\|u\|_B^2,$$

$$\frac{d}{dt}\|u\|_B^2 + 2\tilde{\lambda}_1\|u\|_B^2 \leq 0,$$

$$\frac{\frac{d}{dt}\|u\|_B^2}{\|u\|_B^2} \leq -2\tilde{\lambda}_1,$$

$$\frac{d(\|u\|_B^2)}{\|u\|_B^2} \leq -2\tilde{\lambda}_1 dt.$$

Integrating the result in borders $(0, t)$, the following estimate can be done:

$$\ln\frac{\|u(t)\|_B^2}{\|u_0\|_B^2} \leq -2\tilde{\lambda}_1 t,$$

$$\frac{\|u(t)\|_B^2}{\|u_0\|_B^2} \leq e^{-2\tilde{\lambda}_1 t},$$

$$\|u(t)\|_B^2 \leq \|u_0\|_B^2 e^{-2\tilde{\lambda}_1 t},$$

or

$$\|u(t)\|_B \leq \|u_0\|_B e^{-\tilde{\lambda}_1 t}.$$

Let us get back to our model problem (1). The problem can be written in the form (10) if one lets

$$Bu = [1 + K\delta(x - \xi)]u, \quad (u, v)_B = \int_0^1 u(x)v(x)dx + Ku(\xi)v(\xi).$$

Thus, the following spectral problem can be obtained:

$$-\frac{d^2U}{dx^2} = \lambda[1 + K\delta(x-\xi)]U(x), \quad x \in (0,1)$$

$$U(0) = U(1) = 0,$$

or

$$-\frac{d^2U}{dx^2} = \lambda U(x), \quad x \in (0,\xi) \cup (\xi, 1),$$

$$U(0) = U(1) = 0, \tag{12}$$

$$[U]_{x=\xi} \equiv U(\xi+0) - U(\xi-0) = 0, \quad -\left[\frac{dU}{dx}\right]_{x=\xi} = \lambda K U(\xi).$$

The solution of this problem can be written in the following explicit form:

$$U(x) = \begin{cases} A \sin \alpha x, & x \in (0,\xi) \\ B \sin \alpha(1-x), & x \in (\xi, 1). \end{cases}$$

It is obvious that it satisfies the boundary conditions. The values of the constants A and B can be obtained by the first condition of conjugation, and we get $A = C \sin \alpha(1-\xi)$, $B = C \sin \alpha\xi$, where C is a multiplicative constant, so we can set $C = 1$. From (12), using the second condition of conjugation, we obtain

$$\alpha \sin \alpha\xi \, \cos \alpha(1-\xi) + \alpha \sin \alpha(1-\xi) \, \cos \alpha\xi = \alpha^2 K \sin \alpha\xi \, \sin \alpha(1-\xi),$$

i.e.

$$\alpha = \frac{1}{K}[\operatorname{ctg} \alpha(1-\xi) + \operatorname{ctg} \alpha\xi].$$

If $\xi = 1/2$, the equation takes the following form

$$\alpha = \frac{2}{K} \operatorname{ctg} \frac{\alpha}{2}.$$

Its solutions are α_n, $n = 1, 2, \ldots$. Now, using the condition $\lambda = \alpha^2$, we can obtain the eigenvalues $\lambda_n = \alpha_n^2$, $n = 1, 2, \ldots$. The graphical solution of this problem is shown in the fig. 1, and the numerical values of α_n and λ_n are shown in the table 1.

i	α_i	λ_i
1	1.07687	1.15965
2	3.64360	13.27582
3	6.57833	43.27442
4	9.62956	92.72843
5	12.72230	161.85692

Table 1

In the fig. 2 the first three eigenfunctions U_1, U_2 and U_3 are presented.

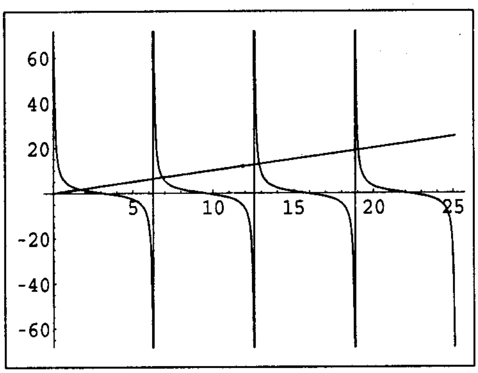

Figure 1

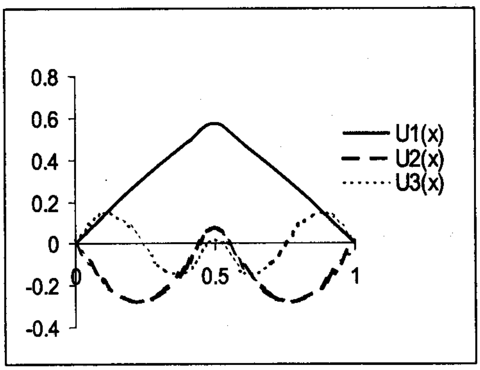

Figure 2

If $\xi \neq 1/2$ we get $\alpha = F(\alpha)$,

$$F(\alpha) = \frac{1}{K}[\operatorname{ctg} \alpha(1-\xi) + \operatorname{ctg} \alpha\xi].$$

It is easy to see that $F(\alpha)$ is a sum of two periodical functions which have different periods. The solutions of a transcendent equation are $\alpha = \alpha_n$, $n = 1, 2, \ldots$. So we can obtain the eigenvalues

$$\lambda = \lambda_n = \alpha_n^2, \quad 0 < \lambda_1 < \lambda_2 \cdots, \; \lambda_n \to \infty,$$

and respective eigenfunctions $U = U_n(x),\ \ n = 1, 2, \ldots$.

If we change the boundary conditions in (1), the different model problem can be obtained:

$$[1 + K\delta(x-\xi)]\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in Q = (0,1) \times (0,T),$$

with boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0,$$

and initial value

$$u(x,0) = u_o(x).$$

The problem can be written in the following form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in (0,\xi) \cup (\xi,1), \ \ t \in (0,T),$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0, \quad u(x,0) = u_0(x),$$

$$\left[\frac{\partial u}{\partial x}\right]_{x=\xi} = K\frac{\partial u}{\partial t}(\xi,t).$$

Using the same procedure as it was done in the first model problem, the following spectral problem can be obtained

$$-\frac{d^2U}{dx^2} = \lambda[1 + K\delta(x-\xi)]U(x), \quad x \in (0,1),$$

$$U(0) = 0, \ \ U'(1) = 0,$$

or

$$-\frac{d^2U}{dx^2} = \lambda U(x), \ \ x \in (0,\xi) \cup (\xi,1)$$

$$U(0) = 0, \quad U'(1) = 0,$$

$$[U]_\xi = 0, \quad -\left[\frac{dU}{dx}\right]_\xi = \lambda K U(\xi).$$

The solution of this spectral problem can be written in explicit form

$$U(x) = \begin{cases} A\sin\alpha x, & x \in (0,\xi) \\ B\cos\alpha(1-x), & x \in (\xi,1). \end{cases}$$

It is obvious that it automatically satisfies the boundary conditions. We obtain the values of the constants A and B using the first condition of conjugation: $A = C \cos \alpha(1-\xi)$, $B = C \sin \alpha\xi$. Here, C is a multiplicative constant, and we can set $C = 1$. By the second condition of conjugation, and the fact that $\lambda = \alpha^2$, we obtain

$$\alpha = \frac{1}{K}[\operatorname{ctg} \alpha\xi) - \operatorname{tg} \alpha(1-\xi)].$$

If $\xi = 1/2$, the equation takes the form $\alpha = \dfrac{2}{K}\operatorname{ctg} \alpha$ and its solutions are α_n, $n = 1, 2, \ldots$.

i	α_i	λ_i
1	1.72076	2.96101
2	6.85124	46.93949
3	12.87460	165.75532
4	19.05870	363.23404
5	31.54250	994.92931

Table 2

Now, using the condition $\lambda = \alpha^2$, we can obtain the eigenvalues $\lambda_n = \alpha_n^2$. $n = 1, 2, \ldots$.

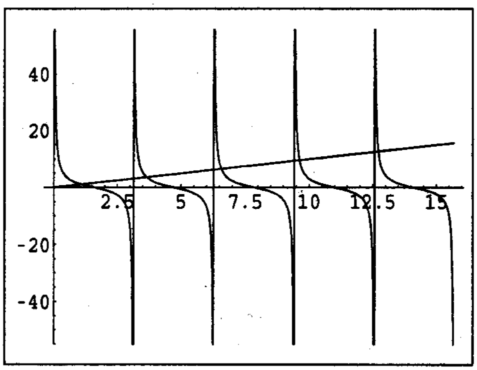

Figure 3

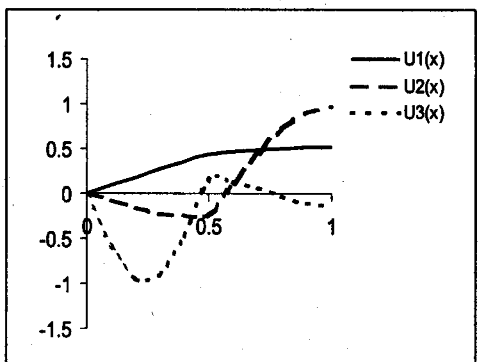

Figure 4

The graphical solution of this problem is shown in the fig. 3, and the numerical values of α_n and λ_n are shown in the table 2. In the fig. 4 the first three eigenfunctions U_1, U_2 and U_3 are presented.

If $\xi \neq 1/2$ we get $\alpha = F(\alpha)$,

$$F(\alpha) = \frac{1}{K}[\operatorname{ctg}\alpha(\xi) - \operatorname{tg}\alpha(1-\xi)].$$

It is easy to see that $F(\alpha)$ is a sum of two periodical functions which have different periods. The solutions of a transcendent equation are $\alpha = \alpha_n$, $n = 1, 2, \ldots$. So we can obtain the eigenvalues

$$\lambda = \lambda_n = \alpha_n^2, \quad 0 < \lambda_1 < \lambda_2 \cdots, \; \lambda_n \to \infty,$$

and respective eigenfunctions $U = U_n(x)$, $n = 1, 2, \ldots$.

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ЕДНА КЛАСА НЕСТАНДАРДНИ СПЕКТРАЛНИ ПРОБЛЕМИ

Соња Геговска-Зајкова

Резиме

Во трудот се разгледувани параболични равенки кои соодветствуваат на равенката на топлопроводност со концентриран капацитет. Изучувани се нестандардни спектрални проблеми во кои сопствените вредности се појавуваат во условите за конјугација.

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