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ON THE SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS FOR THE GEODESIC LINE

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Let the system

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad i=1,2,\dots,n \tag{1}$$

be given, where Γ^i_{jk} are n^3 functions of $x^1,\dots,x^n$.

Lemma. If $x^1(s),\dots,x^n(s)$ is a solution of (1), then it is a solution of the system

$$\frac{d}{\sqrt{dx^r dx^r}}\left(\frac{dx^i}{\sqrt{dx^s dx^s}}\right) + \Gamma^i_{jk}\frac{dx^j}{\sqrt{dx^r dx^r}}\frac{dx^k}{\sqrt{dx^s dx^s}} - \Gamma^\lambda_{\mu\nu}\frac{dx^i}{\sqrt{dx^r dx^r}}\frac{dx^\lambda}{\sqrt{dx^s dx^s}}\frac{dx^\mu}{\sqrt{dx^j dx^j}}\frac{dx^\nu}{\sqrt{dx^k dx^k}} = 0, \quad i=1,2,\dots,n \tag{2}$$

Proof.

$$\frac{d}{\sqrt{dx^r dx^r}}\left(\frac{dx^i}{\sqrt{dx^s dx^s}}\right) + \Gamma^i_{jk}\frac{dx^j}{\sqrt{dx^r dx^r}}\frac{dx^k}{\sqrt{dx^s dx^s}} - \Gamma^\lambda_{\mu\nu}\frac{dx^i}{\sqrt{dx^r dx^r}}\frac{dx^\lambda}{\sqrt{dx^s dx^s}}\frac{dx^\mu}{\sqrt{dx^j dx^j}}\frac{dx^\nu}{\sqrt{dx^k dx^k}} =$$

$$= \frac{d}{\sqrt{\frac{dx^r}{ds}\frac{dx^r}{ds}}\,ds}\left(\frac{\frac{dx^i}{ds}}{\sqrt{\frac{dx^s}{ds}\frac{dx^s}{ds}}}\right) + \Gamma^i_{jk}\frac{\frac{dx^j}{ds}}{\sqrt{\frac{dx^r}{ds}\frac{dx^r}{ds}}}\frac{\frac{dx^k}{ds}}{\sqrt{\frac{dx^s}{ds}\frac{dx^s}{ds}}} -$$

$$- \Gamma^\lambda_{\mu\nu}\frac{\frac{dx^i}{ds}}{\sqrt{\frac{dx^r}{ds}\frac{dx^r}{ds}}}\frac{\frac{dx^\lambda}{ds}}{\sqrt{\frac{dx^s}{ds}\frac{dx^s}{ds}}}\frac{\frac{dx^\mu}{ds}}{\sqrt{\frac{dx^j}{ds}\frac{dx^j}{ds}}}\frac{\frac{dx^\nu}{ds}}{\sqrt{\frac{dx^k}{ds}\frac{dx^k}{ds}}} =$$

$$= \frac{\frac{d^2x^i}{ds^2}\sqrt{\frac{dx^r}{ds}\frac{dx^r}{ds}} - \frac{dx^i}{ds}\frac{dx^s}{ds}\frac{d^2x^s}{ds^2}\cdot\frac{1}{\sqrt{(dx^r/ds)(dx^r/ds)}}}{\left(\sqrt{\frac{dx^s}{ds}\frac{dx^s}{ds}}\right)^3} +$$

$$+\Gamma^i_{jk}\frac{\frac{dx^j}{ds}}{\sqrt{\frac{dx^r}{ds}\frac{dx^r}{ds}}}\frac{\frac{dx^k}{ds}}{\sqrt{\frac{dx^s}{ds}\frac{dx^s}{ds}}}-$$

$$-\frac{\frac{dx^i}{ds}}{\sqrt{\frac{dx^r}{ds}\frac{dx^r}{ds}}}\frac{\frac{dx^\lambda}{ds}}{\sqrt{\frac{dx^s}{ds}\frac{dx^s}{ds}}}\frac{\frac{dx^\mu}{ds}}{\sqrt{\frac{dx^j}{ds}\frac{dx^j}{ds}}}\frac{\frac{dx^\nu}{ds}}{\sqrt{\frac{dx^k}{ds}\frac{dx^k}{ds}}}\Gamma^\lambda_{\mu\nu}$$

Substituting $\frac{d^2x^i}{ds^2}=-\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}$ and $\frac{d^2x^s}{ds^2}=-\Gamma^s_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}$ it is easy to receive that the previous expression is equal to zero.▯

Let $x^1(s),\dots,x^n(s)$ be a solution of the system (1). Then there exist functions $\alpha_1(s),\dots,\alpha_n(s)$ and $\rho(s)$, such that

$$\left.\begin{aligned}\frac{dx^1}{ds}&=\rho\cos\alpha_1\\ \frac{dx^2}{ds}&=\rho\cos\alpha_2\\ &\dots\dots\dots\dots\\ \frac{dx^n}{ds}&=\rho\cos\alpha_n\end{aligned}\right\}\qquad(3)$$

where $\sum_{i=1}^{n}\cos^2\alpha_i=1$. Substituting $\frac{dx^i}{ds}$ from (3) in (2), we get

$$\left.\begin{aligned}\frac{d\cos\alpha_i}{\rho ds}+\Gamma^i_{jk}\cos\alpha_j\cos\alpha_k-\cos\alpha_i\cos\alpha_\lambda\cos\alpha_\mu\cos\alpha_\nu\Gamma^\lambda_{\mu\nu}=0\\ i=1,2,\dots,n\end{aligned}\right\}\qquad(4)$$

and substitutitng (3) in (1)

$$\frac{d(\rho\cos\alpha_i)}{ds}+\Gamma^i_{jk}\,\rho^2\cos\alpha_j\cos\alpha_k=0,\quad i=1,2,\dots,n\qquad(5)$$

Now, for $i=1,\dots,n$ it is $\Gamma^i_{jk}\cos\alpha_j\cos\alpha_k=-\frac{d(\rho\cos\alpha_i)}{\rho^2ds}$. Substituting this in (4) we receive

$$\frac{d\cos\alpha_i}{\rho ds} - \frac{d(\rho\cos\alpha_i)}{\rho^2 ds} - \cos\alpha_i(\cos\alpha_\lambda\cos\alpha_\mu\cos\alpha_\nu\Gamma^\lambda_{\mu\nu}) = 0$$

$$\cos\alpha_i\left(\frac{d\rho}{\rho^2 ds} + \cos\alpha_\lambda\cos\alpha_\mu\cos\alpha_\nu\Gamma^\lambda_{\mu\nu}\right) = 0.$$

There exists an index i such that $\cos\alpha_i \neq 0$, so

$$\frac{d(\frac{1}{\rho})}{ds} = \cos\alpha_\lambda\cos\alpha_\mu\cos\alpha_\nu\Gamma^\lambda_{\mu\nu}. \tag{6}$$

We introduce a new parameter ℓ by the equation

$$d\ell = \sqrt{dx^r\, dx^r} \tag{7}$$

or, by (3)

$$d\ell = \rho ds. \tag{8}$$

The system (2) becomes

$$\left.\begin{array}{c} \dfrac{d^2x^i}{d\ell^2} + \Gamma^i_{jk}\dfrac{dx^j}{d\ell}\dfrac{dx^k}{d\ell} - \dfrac{dx^i}{d\ell}\dfrac{dx^\lambda}{d\ell}\dfrac{dx^\mu}{d\ell}\dfrac{dx^\nu}{d\ell}\Gamma^\lambda_{\mu\nu} = 0 \\ i=1,2,\dots,n \end{array}\right\} \tag{9}$$

As $\frac{dx^i}{d\ell}\frac{dx^i}{d\ell} = 1$, system (9) is equivalent to

$$\left.\begin{array}{c} \dfrac{d^2x^i}{d\ell^2} + \Gamma^i_{jk}\dfrac{dx^j}{d\ell}\dfrac{dx^k}{d\ell} - \dfrac{dx^i}{d\ell}\dfrac{dx^\lambda}{d\ell}\dfrac{dx^\mu}{d\ell}\dfrac{dx^\nu}{d\ell}\Gamma^\lambda_{\mu\nu} = 0 \\ i=1,2,\dots,n-1 \\ \left(\dfrac{dx^1}{d\ell}\right)^2 + \dots + \left(\dfrac{dx^n}{d\ell}\right)^2 = 1. \end{array}\right\} \tag{10}$$

System (10) becomes a system of n-1 equations with n-1 unknowns functions $x^1(\ell),\dots,x^{n-1}(\ell)$.

Let $x^1(\ell),\dots,x^n(\ell)$ be a solution for the system (10). It is obvious that $\frac{dx^i}{d\ell} = \cos\alpha_i$ $(i=1,\dots,n)$. Componentes Γ^i_{jk} are functions of ℓ and the left side of (6) is a known function of ℓ. From (6) and (8) we get

$$\frac{d(\frac{1}{\rho})}{\frac{1}{\rho}\, d\ell} = \frac{dx^\lambda}{d\ell}\frac{dx^\mu}{d\ell}\frac{dx^\nu}{d\ell}\Gamma^\lambda_{\mu\nu}$$

$$\frac{d\ln\rho}{d\ell} = -\frac{dx^\lambda}{d\ell}\frac{dx^\mu}{d\ell}\frac{dx^\nu}{d\ell}\Gamma^\lambda_{\mu\nu}$$

$$\rho = \exp(-\int\frac{dx^\lambda}{d\ell}\frac{dx^\mu}{d\ell}\frac{dx^\nu}{d\ell}\Gamma^\lambda_{\mu\nu}d\ell),$$

From (8) we receive

$$s = \int\exp(\int\frac{dx^\lambda}{d\ell}\frac{dx^\mu}{d\ell}\frac{dx^\nu}{d\ell}\Gamma^\lambda_{\mu\nu}\, d\ell)d\ell. \quad (11)$$

The inverse function of the function (11), expresses ℓ as a function of s. So we know $x^1,\dots,x^n$ as functions of s and the system (1) is solved. At the end we can summary the result: The solving of the system (1) reduces to the system (10), and the two integrals from the right side of the equation (11).

R E F E R E N C E S

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ЗА РЕШЕНИЕТО НА СИСТЕМОТ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ ЗА ГЕОДЕЗИСКИТЕ ЛИНИИ

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Резиме

Нека е даден системот диференцијални равенки (1). Се покажува дека ако $x^1(s),\dots,x^n(s)$ е решение на системот (1) тогаш тоа е решение и на системот (2). Користејќи го тоа, решавањето на системот (1) се сведува на решавање на системот (10), кој може да се третира како систем од n-1 равенки со n-1 непознати функции. Врската меѓу параметрите s и ℓ е дадена со релацијата (11).