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NEW FORMULAR FOR APPROXIMATE SOLUTIONS OF THE LINEAR DIFFERENTIAL EQUATIONS OF THE II AND III ORDER

M. Kujumdzieva-Nikoloska, D. Dimitrovski

Abstract. In this paper is used a simple quadrature process for approximate solution of a linear differential equations of II and III order with analytical coefficients.

I. CANONICAL FORM. We shall prove that following

Theorem. Equation

$$y'' + a(x)y = 0 \tag{1}$$

has linear independent solutions

$$y_1 = 1-\int_0^x\int_0^x a(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2 - \dots \tag{2}$$

$$y_2 = x-\int_0^x\int_0^x xa(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x xa(x)dx^2 - \dots \tag{2'}$$

Proof. If we differentiate (2) two times we get

$$y_1'' = -a(x)+a(x)\int_0^x\int_0^x a(x)dx^2 - \dots = -a(x)[1-\int_0^x\int_0^x a(x)dx^2+\dots] =$$
$$= -a(x)y_1$$

and similarly for y_2.

If the function a(x) is such that we can easily calculate double, quater etc. Integrals, then we can also find y_1 and y_2 quickly with great accuracy.

If the quadratures are dificult, we must solve (1) approximately taking finite numbers of (2) (or (2')) or with approximation of a(x).

It is evident that the following essential and elementary statements hold:

Theorem. Series (2) and (2') are convergent for every analytical coefficient a(x).

Proof. With a usual estimation, if $|a(x)| < M$, then

$$|y_1| < 1 + M\int_0^x\int_0^x dx^2 + M^2\int_0^x\int_0^x dx^2\int_0^x\int_0^x |x| dx^2 + \ldots = \sum_{n=0}^{\infty} \frac{[(\sqrt{M}x)^2]^n}{(2n)!} \quad (3)$$

$$|y_2| < |x| + M\int_0^x\int_0^x |x| dx^2 + M^2\int_0^x\int_0^x dx^2\int_0^x\int_0^x |x| dx^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{\sqrt{M}} \frac{(\sqrt{M}|x|)^{2n+1}}{(2n+1)!} \quad (3')$$

and these series are convergent for every x and if $|x| \le x_0$ belongs to the analytical domain, for solution y_1 we estimate the error from

$$|y_1| \approx 1 + M\frac{x^2}{2!} + M^2\frac{x^4}{4!} + \ldots + M^n \frac{x^{2n}}{(2n)!} + R_n.$$

By the Taylor series we get

$$R_n < \frac{(\sqrt{M}x_0)^{2n}}{(2n)!} \frac{\frac{x_0^2}{(2n+1)(2n+2)}}{1 - Mx_0^2} \quad \text{for } Mx_0^2 < 1, \quad (4)$$

and because estimating series is fast convergent then the error is relatively small. It is posible for given x_0 and M, to find, in usual way, the number of the terms n, in order to make the error less than ε.

Similarely, for

$$|y_2| \approx |x| + M\frac{|x|^3}{3!} + \ldots + M^n \frac{|x|^{2n+1}}{(2n+1)!} + R_n$$

and if $|x| \le x_0$

$$R_n < \frac{(\sqrt{M}x_0)^{2n}}{(2n+3)!} \frac{M\sqrt{M}x_0^3}{1 - Mx_0^2} \quad (4')$$

for $Mx_0^2 < 1$.

Example 1. For differential equation

$$y'' + \frac{x^2}{(1+x^2)^3} y = 0$$

$$0 \le a(x) = \frac{x^2}{(1+x^2)^3} \le \frac{4}{27} \text{ and}$$

$$|y_1| = |u_{10} - u_{11} + u_{12} - \ldots + (-1)^n u_{1n} + \ldots| =$$

$$= |1 - \frac{1}{8}[\frac{1}{1+x^2} + x\,\mathrm{arctg}x - 1] + \frac{1}{8}[\frac{1}{24}\frac{1}{(1+x^2)^2} - \frac{5}{24}\frac{1}{1+x^2} -$$

$$- \frac{3}{16}(\mathrm{arctg}x^2)] + \frac{1}{8}[\frac{x\,\mathrm{arctg}x}{1+x^2} + \frac{1}{12}x\,\mathrm{arctg}x + \frac{13}{96}\ln(1+x^2) -$$

$$- \frac{19}{96}] - \ldots| \le \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!\,3^{3n}}$$

and

$$|y_2| = |u_{20} - u_{21} + u_{22} - \ldots + (-1)^n u_{2n} + \ldots| =$$

$$= |x - [-\frac{3}{8}\text{arctg}x + \frac{1}{8}\frac{x}{1+x^2} + \frac{1}{4}x] +$$

$$+ [-\frac{3}{64}\frac{1}{1+x^2}\text{arctg}x - \frac{13}{192}\text{arctg}x - \frac{3}{128}\text{aarctg}^2x +$$

$$+ \frac{15}{256}\frac{x}{1+x^2} + \frac{1}{192}\frac{x}{(1+x^2)^2}] - \ldots| < \sum_{n=0}^{\infty}\frac{1}{2}\frac{(2|x|)^{2n+1}}{(2n+1)!3^{2n}}$$

Example 2. For $y'' + \frac{1}{e^{x^2}}y = 0$

$$e^{-x^2} \cong 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \ldots + (-1)^n\frac{x^{2n}}{n!} + r_n$$

$$|a(x)| \le 1 \text{ and}$$

$$|y_1| = |1 - \sum_{k=0}^{n}\frac{(-1)^k x^{2k+2}}{(k+1)!2(2k+1)} + \ldots| \le \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

$$|y_2| = |x - \sum_{k=0}^{n}\frac{(-1)^k x^{2k+3}}{(k+1)!2(2k+3)} + \ldots| \le \sum_{n=0}^{\infty}\frac{|x|^{2n+1}}{(2n+1)!}$$

II. THE GENERAL LINEAR DIFFERENTIAL EQUATIONS OF THE II ORDER. In a similar way, we can also deal with the general linear differential equation of the II order

$$y'' + a(x)y' + b(x)y = 0 \tag{5}$$

Theorem. The general solution of (5) is given by

$$y = e^{-\frac{1}{2}\int_0^x a(x)dx}\{C_0[1 - \int_0^x\int_0^x(b - \frac{a'}{2} - \frac{a^2}{4})dx^2 + \int_0^x\int_0^x(b - \frac{a'}{2} - \frac{a^2}{4})dx^2$$

$$\int_0^x\int_0^x(b - \frac{a'}{2} - \frac{a^2}{4})dx^2 - \ldots] + C_1[x - \int_0^x\int_0^x x(b - \frac{a'}{2} - \frac{a^2}{4})dx^2 + \tag{6}$$

$$+ \int_0^x\int_0^x(b - \frac{a'}{2} - \frac{a^2}{4})dx^2\int_0^x\int_0^x x(b - \frac{a'}{2} - \frac{a^2}{4})dx^2 - \ldots]\}$$

Proof. With substitution $y = e^{-1/2\int a(x)dx}z$, where z is a new unknown function the equation (5) can be transformed in the canonical form (1)

$$z'' + A(x)z = 0$$

where $A(x) = b(x) - \frac{a'(x)}{2} - \frac{a^2(x)}{4}$, and consequently the integrals (2) and (2') hold for z.

However, in the differential equations the following principle is satisfied: every solution depends only on the coefficients. Therefore the solution of (5) will depend only on coefficients a(x) and b(x), i.e.

$$y = f(a(x), b(x)) \tag{7}$$

So, from (6) we get:

<u>Theorem</u>. Every particular solution of the linear homogenious differential equation of the II order (6) can be expressed by a sum of 3 factors, done with the formula

$$y = e^{-1/2\int a(x)dx}[Y_a + Y_b + Y_{a,b}] \tag{8}$$

where Y_a is a part of the solution which depends only on coefficient a(x), Y_b depends only on b(x), and $Y_{a,b}$ depends on the total influence on a(x) and b(x) in the solution. So for y_1

$$\begin{aligned}
Y_b &= 1 - \int_0^x\!\!\int_0^x b(x)dx^2 + \int_0^x\!\!\int_0^x b(x)dx^2 \int_0^x\!\!\int_0^x b(x)dx^2 - \\
&\quad - \int_0^x\!\!\int_0^x b(x)dx^2 \int_0^x\!\!\int_0^x b(x)dx^2 \int_0^x\!\!\int_0^x b(x)dx^2 + \dots \\
Y_a &= \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 + \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 + \\
&\quad + \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 + \dots \\
Y_{a,b} &= -\int_0^x\!\!\int_0^x b dx^2 \quad \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 - \\
&\quad - \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \quad \int_0^x\!\!\int_0^x b dx^2 + \int_0^x\!\!\int_0^x b dx^2 \int_0^x\!\!\int_0^x b dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 + \\
&\quad + \int_0^x\!\!\int_0^x b dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x b dx^2 - \\
&\quad - \int_0^x\!\!\int_0^x b dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 - \\
&\quad - \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x b dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 - \\
&\quad - \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x \left(\frac{a'}{2} + \frac{a^2}{4}\right)dx^2 \int_0^x\!\!\int_0^x b dx^2 + \dots
\end{aligned} \tag{9}$$

Analogicaly we get expresion for y_2. So we get one new practical formula.

Theorem. If a(x) and b(x) are analytical coefficients then the following approximate formula for the solution of equation (6) holds:

$$y \approx e^{-1/2\int a(x)dx}\left(1-\int_0^x\int_0^x \left(b(x) - \frac{a'(x)}{2} - \frac{a^2(x)}{4}\right)dx\right) \qquad (10)$$

whose accuracy can be easily estimated.

Proof. If a(x) and b(x) are analytical coefficients, i.e. limited and if

$$m_0 \le |a(x)| \le m_1$$

then

$$M_0 \le \left|b(x) - \frac{a'(x)}{2} - \frac{a^2(x)}{4}\right| \le M_1$$

$$|y_1| < e^{-1/2\cdot m_0|x|}\left(1+\int_0^x\int_0^x M_1 dx^2 + \int_0^x\int_0^x M_1 dx^2\int_0^x\int_0^x M_1 dx^2 + \ldots\right)$$

$$= e^{-1/2\cdot m_0|x|}\left(1+M_1\frac{x^2}{2!}+M_1^2\frac{x^4}{4!} + \ldots\right)$$

and

$$|y_1| \approx e^{-1/2\cdot m_0|x|}\left(1+M_1\frac{x^2}{2!}+M_1^2\frac{x^4}{4!} + \ldots + M_1^n\frac{x^{2n}}{(2n)!} + R_n\right)$$

Similarly

$$|y_2| \approx e^{-1/2\cdot m_0|x|}\left(|x|+M_1\frac{|x|^3}{3!} + M_1^2\frac{|x|^5}{5!} + \ldots + \frac{M_1|x|^{2n+1}}{(2n+1)!} + R_n\right)$$

and so we get the approximative formula (10).

Example. For

$$y'' + y' + \cos x\, y = 0$$

$$a(x) = 1, \quad b(x) = \cos x, \quad |a| = 1$$

$$0 \le \left|b(x) - \frac{a'(x)}{2} + \frac{a^2(x)}{4}\right| \le \left|\cos x - \frac{1}{4}\right| \le \frac{5}{4}$$

One approximative solution is

$$y^* = e^{-1/2\int a(x)dx}\left[1-\int_0^x\int_0^x\left(\cos x-\frac{1}{4}\right)dx^2\right] = e^{-x/2}\left(\cos x+\frac{1}{8}x^2\right).$$

III. CANONICAL EQUATION OF THE III ORDER. We can give a similar observation for the differential equation of the III order.

Theorem. The canonical differential equation of the III order

$$y''' + a(x)y = 0 \qquad (11)$$

has particular integrals

$$y_1 = 1 - \int_0^x\int_0^x\int_0^x a(x)dx^3 + \int_0^x\int_0^x\int_0^x a(x)dx^3\int_0^x\int_0^x\int_0^x a(x)dx^3 - \dots$$

$$y_2 = x - \int_0^x\int_0^x\int_0^x xa(x)dx^3 + \int_0^x\int_0^x\int_0^x a(x)dx^3\int_0^x\int_0^x\int_0^x xa(x)dx^3 - \dots \qquad (12)$$

$$y_3 = x^2 - \int_0^x\int_0^x\int_0^x x^2a(x)dx^3 + \int_0^x\int_0^x\int_0^x a(x)dx^3\int_0^x\int_0^x\int_0^x x^2a(x)dx^3 - \dots$$

Proof. We can prove it directly, by differentiation:

Theorem. Series (12) are convergent for every analytical coefficient a(x).

Proof. From $|a(x)| < M$ we have that

$$|y_1| < 1 + M\frac{|x|^3}{3!} + M^2\frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(\sqrt[3]{M}|x|)^{3n}}{(3n)!}$$

$$|y_2| < |x| + M\frac{x^4}{4!} + M^2\frac{|x|^6}{7!} + \dots = \sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{M}} \frac{(\sqrt[3]{M}|x|)^{3n+1}}{(3n+1)!}$$

$$|y_3| < x^2 + M\frac{|x|^5}{5!} + M\frac{2x^8}{8!} + \dots = x^2 + \sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{M^2}} \frac{(\sqrt[3]{M}|x|)^{3n+2}}{(3n+2)!}$$

and because the estimating series are convergent for every x, the same holds for series (12) for every $|x| \le x_0$ from the domain where a(x) is an analytical function. In this way for the approximative solutions we can also estimate the error.

IV. A MORE GENERAL EQUATION OF THE III ORDER. For the homogenous linear differential equation of the III order with two analytical coefficients

$$y''' + a(x)y' + b(x)y = 0 \qquad (13)$$

using Cauchy's method of the unknown coefficients for solwing with series, by putting

$$a(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad b(x) = \sum_{n=0}^{\infty} b_n x^n$$

and finding unknown series $y = \sum_{n=0}^{\infty} c_n x^n$, we get:

$$
\begin{aligned}
y_1 = 1 &- \int_0^x\int_0^x\int_0^x b(x)\,dx^3 + \int_0^x\int_0^x\int_0^x a(x)\,dx^3\int_0^x\int_0^x b(x)\,dx^2 + \qquad (14)\\
&+ \int_0^x\int_0^x\int_0^x b(x)\,dx^3\int_0^x\int_0^x\int_0^x b(x)\,dx^3 - \int_0^x\int_0^x\int_0^x a(x)\,dx^3\int_0^x\int_0^x a(x)\,dx^2\int_0^x\int_0^x b(x)\,dx^2 -\\
&- \int_0^x\int_0^x\int_0^x a(x)\,dx^3\int_0^x\int_0^x b(x)\,dx^2\int_0^x\int_0^x\int_0^x b(x)\,dx^3 - \int_0^x\int_0^x\int_0^x b(x)\,dx^3\int_0^x\int_0^x a(x)\,dx^2\,\cdot\\
&\int_0^x\int_0^x b(x)\,dx^2 - \int_0^x\int_0^x\int_0^x b(x)\,dx^3\int_0^x\int_0^x\int_0^x b(x)\,dx^3\int_0^x\int_0^x\int_0^x b(x)\,dx^3 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
y_2 = x &- \int_0^x\int_0^x\int_0^x a\,dx^3 - \int_0^x\int_0^x\int_0^x xb\,dx^3 + \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2 + \qquad (15)\\
&+ \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x xb\,dx^3 + \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3 + \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x xb\,dx^3 -\\
&- \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x a\,dx^2 - \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x xb\,dx^2 -\\
&- \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x\int_0^x a\,dx^2 - \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x\int_0^x xb\,dx^3 -\\
&- \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2 - \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x xb\,dx^2 - \ldots
\end{aligned}
$$

$$
\begin{aligned}
y_3 = x^2 &- 2\int_0^x\int_0^x\int_0^x ax\,dx^3 - \int_0^x\int_0^x\int_0^x x^2b\,dx^3 + \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x xa\,dx^2 + \qquad (16)\\
&+ \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x x^2b\,dx^2 + 2\int_0^x\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x\int_0^x xa\,dx^3 + \int_0^x\int_0^x\int_0^x b\int_0^x\int_0^x\int_0^x x^2b\,dx^3 -\\
&- 2\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x xa\,dx^2 - \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x x^2b\,dx^2 -\\
&- 2\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x xa\,dx^2 - \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x\int_0^x x^2b\,dx^3 -\\
&- 2\int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x xa\,dx^2 - \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x x^2b\,dx^2 - \ldots
\end{aligned}
$$

So we get

<u>Theorem</u>. The linear homogenous differential equation of the III order (13) has particular solution in the forme of series of integrals which depend on coefficients a(x) and b(x) given with (14), (15) and (16).

It can be used for creating a practical theorem for an approximate calculation of particular solutions.

<u>Theorem</u>. Every solution of linear homogenous differential equation of III order (13) with analytical coefficients is givei by the sum:

$$y(x) = Y_a + Y_b + Y_{a,b} \tag{17}$$

where Y_a is a series of integrals which depends only on coefficient $a(x)$, Y_b is a series of integrals which depends only on coefficients $b(x)$ and $Y_{a,b}$ is a series of integrals which depends on $a(x)$ and $b(x)$, and they are for (15):

$$Y_b = x - \int_0^x\!\int_0^x\!\int_0^x xb\,dx^3 + \int_0^x\!\int_0^x\!\int_0^x b\,dx^3\int_0^x\!\int_0^x\!\int_0^x xb\,dx^3 - \int_0^x\!\int_0^x\!\int_0^x b\,dx^3\int_0^x\!\int_0^x\!\int_0^x b\,dx^3\int_0^x\!\int_0^x\!\int_0^x xb\,dx^3 + \dots$$

$$Y_a = -\int_0^x\!\int_0^x\!\int_0^x a\,dx^3 + \int_0^x\!\int_0^x\!\int_0^x x\,dx^3\int_0^x\!\int_0^x a\,dx - \int_0^x\!\int_0^x\!\int_0^x a\,dx^3\int_0^x\!\int_0^x a\,dx^2 + \dots =$$

$$= \int_0^x\!\int_0^x\!\int_0^x a\,dx^3\left(1-\int_0^x\!\int_0^x a\,dx^2+\int_0^x\!\int_0^x a\,dx^2\int_0^x\!\int_0^x a\,dx^2-\int_0^x\!\int_0^x a\,dx^2\int_0^x\!\int_0^x a\,dx^2\int_0^x\!\int_0^x a\,dx^2 + \dots\right) \tag{18}$$

$$Y_{a,b} = \int_0^x\!\int_0^x\!\int_0^x a\,dx^3\int_0^x\!\int_0^x xb\,dx^2 + \int_0^x\!\int_0^x\!\int_0^x b\,dx^3\int_0^x\!\int_0^x\!\int_0^x a\,dx^3 - \int_0^x\!\int_0^x\!\int_0^x a\,dx^3\int_0^x\!\int_0^x a\,dx^2\int_0^x\!\int_0^x xb\,dx^2 -$$

$$-\int_0^x\!\int_0^x\!\int_0^x a\,dx^3\int_0^x\!\int_0^x b\,dx^2\int_0^x\!\int_0^x\!\int_0^x xb\,dx^3 + \dots$$

Analogous we get also for (14) and (16). So we get practical rules:

<u>Theorem</u>. The expression

$$y^*(x) = x - \int_0^x\!\int_0^x\!\int_0^x xb\,dx^3 + \int_0^x\!\int_0^x\!\int_0^x a\,dx^3 \tag{19}$$

is an appropriate solution for (15) with accuracy to terms of x^4.

<u>Proof</u>. Because $a(x)$ and $b(x)$ are analytical functions, they are limited:

$$|a(x)| < m, \ |b(x)| < m \quad \text{for} \quad |x| \le x_o,$$

i.e.

$$|y^*(x)| \le x_o + \left|\int_0^x\!\int_0^x\!\int_0^x a(x)\,dx^3\right| + \left|\int_0^x\!\int_0^x\!\int_0^x b(x)\,dx^3\right| \le x_o + m\frac{x_o}{3!} + \frac{x_o}{4!}$$

Because y^* is part of (15), and because the first following term we can estimate with:

$$\left|\int_0^x\!\int_0^x\!\int_0^x a\,dx^3\int_0^x\!\int_0^x a\,dx^2\right| < m^2\left|\int_0^x\!\int_0^x\!\int_0^x\!\int_0^x\!\int_0^x dx^5\right| < m^2\frac{x_o}{5!}$$

and all other terms are with greather order of x respectively x_o.

Theorem. The expression

$$y^{**}(x) = x - \int_0^x\int_0^x\int_0^x a\,dx^3 - \int_0^x\int_0^x\int_0^x xb\,dx^3 + \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2 + \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x xb\,dx^2 + \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3 \qquad (20)$$

is an appropriate solution of (15) with the exactness to term x^6.

Proof. If the series of integrals of the solution is substituted by the polynom of integrals, holds the usual estimation of the error for power series. If we take

$$y(x) \approx \Sigma \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{k} a\,dx^k \underbrace{\int_0^x\int_0^x\cdots\int_0^x}_{m} b\,dx^m$$

then the error of the formula can be done by

$$R_n \le \frac{|a|^k|b|^n}{(k+1)!}\,\frac{x_0^{k+m+1}}{1-x_0}, \quad \text{for } |x| \le x_0 < 1.$$

Similarly, if we need a greater precision, we can take more terms of (15).

Theorem. The expression

$$\begin{aligned} y^{***} = x &- \int_0^x\int_0^x\int_0^x xb\,dx^3 - \int_0^x\int_0^x\int_0^x a\,dx^3 + \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2 + \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x xb\,dx^2 + \\ &+ \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3 + \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x xb\,dx^3 - \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x a\,dx^2 - \\ &- \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2\int_0^x\int_0^x xb\,dx^2 - \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x\int_0^x a\,dx^3 - \qquad (21) \\ &- \int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x b\,dx^2\int_0^x\int_0^x\int_0^x xb\,dx^3 - \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x a\,dx^2 - \\ &- \int_0^x\int_0^x\int_0^x b\,dx^3\int_0^x\int_0^x\int_0^x a\,dx^3\int_0^x\int_0^x xb\,dx^2 \end{aligned}$$

is an appropriate solution of (15) with exactness of $\frac{m^3x^{10}}{10!}$ for $|x| < 1$, and $m = \max(|a(x)|,|b(x)|)$.

Proof. It is similar to the above proof with simple estimation of the integrals.

V. GENERAL EQUATION OF THE III ORDER. All said above is valid for general equation of the III order:

$$y''' + a_1(x)y'' + b_1(x)y' + a(x)y = 0 \qquad (22)$$

It is because with substitution

$$y = e^{-1/3\int a\,(x)dx} z$$

it can be transformed to

$$z''' + a(x)z' + b(x)z = 0 \qquad (23)$$

where z is a new unknown function. The coefficients a(x) and b(x) in (23) depend on $a_1(x)$, $b_1(x)$ and $c_1(x)$ in known manner, so the formulae (17), (18), (19), (20), (21) can be used in (23).

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НОВИ ФОРМУЛИ ЗА ПРИБЛИЖНО РЕШАВАЊЕ НА ЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ ОД II И III РЕД

М. Кујумџиева-Николоска, Д. Димитровски

Р е з и м е

Во овој труд се користи прост квадратурен процес за приближно ешавање на линеарни диференцијални равенки од II и III ред со налитички коефициенти. При тоа се добиени некои формули за нивно риближно решавање и оцeнета е нивната точност.

arija Kujumdzieva-Nikoloska
lektrotehnički fakultet
.f. 574
1000 Skopje
acedonia

Dragan Dimitrovski
Prirodno-matematički fakultet
p.f. 162
91000 Skopje
Macedonia