

CONGRUENCES ON n -GROUPS

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Abstract

A description of congruences on polyadic groups is given in this paper. Namely, if $\mathbf{Q} = (Q, [])$ is an $n + 1$ -group, and \mathbf{Q}^\wedge is its universal covering group, then each congruence α of \mathbf{Q} can be characterized by an invariant subgroup \mathbf{H}_α of \mathbf{Q}^\wedge such that $\mathbf{H}_\alpha \subseteq \{a_1 \dots a_{n-1} \mid a_n \in Q\}$. Also, if $a \in Q$ and $*$ is an operation on Q defined by $x * y = [xa^{n-2}y]$, then $(Q, *)$ is a group, and each congruence α of \mathbf{Q} is characterized by an invariant subgroup K of $(Q, *)$, such that for each $x \in Q$ $[xa^{n-2}K] = [a^{n-2}Kx]$. It is also shown that it may happen to have an n -group \mathbf{Q} and a congruence α on Q such that neither of the α -equivalence classes is an n -subgroup of \mathbf{Q} , and necessary and sufficient conditions are given under which such classes do exist.

In the last section of this paper new and shorter proofs of some propositions of [5] are given using the universal covering group.

0. Let us first state some preliminary definitions and results.

If $[] : (x_0, x_1, \dots, x_n) \mapsto x_0 x_1 \dots x_n$ is an associative $n + 1$ -ary operation on a non empty set Q , then we say that $\mathbf{Q} = (Q; [])$ is an $n + 1$ -semigroup.

The notions of $n + 1$ -subsemigroup, homomorphism, congruence on an $n + 1$ -semigroup ¹⁾ are defined in the usual way, so we will not state them explicitly.

If \mathbf{Q} is an $n + 1$ -semigroup, then the semigroup \mathbf{Q}^\wedge , given by the following presentation (in the variety of all semigroups)

$$\mathbf{Q}^\wedge = \langle Q; \{a = a_0 a_1 \dots a_n \mid a = [a_0 a_1 \dots a_n]\} \rangle \quad (0.1)$$

is called the universal covering semigroup of \mathbf{Q} . We can assume that $Q \subseteq \mathbf{Q}^\wedge$ and

$$Q^\wedge = Q \cup Q^2 \cup \dots \cup Q^n$$

where $Q^i = \underbrace{Q Q \dots Q}_i = \{a_0, a_1 \dots a_i \mid a_i \in Q\}$, and that the union is

disjoint.

¹⁾ In this paper, only for technical reasons, we are using $n + 1$ -semigroups instead of n -semigroups.

An $n + 1$ -semigroup $\mathbf{Q} = (Q; [])$ is said to be an $n + 1$ -group iff
 $(\forall a_0, a_1, \dots, a_{n-1}, b \in Q)(\exists x, y \in Q)[a_0 a_1 \dots a_{n-1} x] = b = [y a_0 a_1 \dots a_{n-1}]$.

There are several axiom systems of $n + 1$ -groups (see, for example, [1]), and we note that an $n + 1$ -semigroup is an $n + 1$ -group iff its universal covering semigroup is a group ([1]).

A homomorphism φ from an $n + 1$ -semigroup \mathbf{Q} into an $n + 1$ -semigroup \mathbf{Q}' can be uniquely extended to a homomorphism $\varphi^\wedge: \mathbf{Q}^\wedge \rightarrow \mathbf{Q}'^\wedge$, such that φ is the restriction of φ^\wedge on \mathbf{Q} . Namely, φ^\wedge is defined in the following way:

$$a = a_1 \dots a_i \in Q^i, \quad a_v \in Q \Rightarrow \varphi^\wedge(a) = \varphi(a_1) \dots \varphi(a_i).$$

We note that if φ is surjective or bijective, then φ^\wedge has the corresponding property, but it can happen φ to be injective mapping and φ^\wedge not to be such one ([4]). Thus, in general, if P is an $n + 1$ -subsemigroup of an $n + 1$ -semigroup \mathbf{Q} , P^\wedge may not be an $n + 1$ -subsemigroup of \mathbf{Q}^\wedge . But if we consider $n + 1$ -groups the following statement holds:

if H is an $n + 1$ -subgroup of the $n + 1$ -group \mathbf{Q} , then H^\wedge is a subgroup of \mathbf{Q}^\wedge ([4]).

1. Let $\mathbf{Q} = (Q; [])$ be an $n + 1$ -semigroup, and \mathbf{Q}^\wedge its universal covering semigroup. Let α be a congruence on \mathbf{Q} . Then $\varphi = \text{nat } \alpha$ is an epimorphism from \mathbf{Q} into \mathbf{Q}/α , and $\varphi^\wedge = (\text{nat } \alpha)^\wedge$ is an epimorphism from \mathbf{Q}^\wedge into $(\mathbf{Q}/\alpha)^\wedge$. If $\lambda^\wedge = \ker(\text{nat } \alpha)^\wedge$, then α^\wedge is a congruence on the semigroup \mathbf{Q}^\wedge with the property

$$x_1 \dots x_i \alpha^\wedge y_1 \dots y_j \Rightarrow i = j \pmod{n}, \quad (x_v, y_\lambda \in Q)^2$$

1.1°. Let \mathbf{Q} be an $n + 1$ -semigroup, and \mathbf{Q}^\wedge its universal covering semigroup. If β is a congruence on \mathbf{Q}^\wedge , then $\alpha = \beta|_Q$ is a congruence on \mathbf{Q} , such that $\alpha^\wedge \subseteq \beta$. \diamond

1.2°. Let \mathbf{Q} be an $n + 1$ -semigroup, and \mathbf{Q}^\wedge its universal covering semigroup. Then β is a congruence on \mathbf{Q}^\wedge with the property

$$\alpha = \beta|_Q \ \& \ x_1 \dots x_i \beta y_1 \dots y_j \Rightarrow i = j \text{ iff } a^\wedge = \beta. \quad \diamond$$

2. Let us now give a description of the subgroups of the universal covering group \mathbf{Q}^\wedge of an $n + 1$ -group \mathbf{Q} .

2.1°. H is a subgroup of \mathbf{Q}^\wedge iff there exists a natural number r , $1 \leq r \leq n$, such that $r|n$, and $H = H_r \cup H_r^2 \cup \dots \cup H_r^q$, where $H_r = H \cap Q^r$, $n = rq$. In this case H_r is a $q + 1$ -subgroup of H and $H_r^\wedge = H([3])$. \diamond

2.2°. H is an invariant subgroup of \mathbf{Q}^\wedge iff for each x in Q , $xH_r = H_r x([3])$. \diamond

Let us note another property of \mathbf{Q}^\wedge . Namely, if $a \in Q$, then $Q^\wedge = Q \cup aQ \cup \dots \cup a^{n-1}Q$, i.e. $Q^i = a^{i-1}Q$. Then $(a^i x)(a^j y) = a^{i+j+1}z$, where z is the solution of the equation $[a^n z] = [a^{n-j-1} x a^j y]$, and $i + j + 1$ is counted by modulo n . Specially, $(a^{n-1}x)(a^{n-1}y) = a^{n-1}z$, where $z = [x a^{n-1}y]$.

2) Here, and further on, we will assume that $x_v, y_\lambda \in Q$ and $i, j \leq n$.

3. Let us now state some properties of congruences of $n + 1$ -groups. Note that, further on with \mathbf{Q} will be denoted an $n + 1$ -group, and with \mathbf{Q}^\wedge its universal covering group. We will often write „subgroup” instead of „ $n + 1$ -subgroup” whenever the meaning will be clear from the context.

3.1°. Let α be a congruence on \mathbf{Q} . Then $(Q/\alpha)^\wedge \cong Q_{/\alpha^\wedge}^\wedge$, and

$$x_1 \dots x_i \alpha^\wedge y_1 \dots y_j \Leftrightarrow i = j \& (\exists c_0, \dots, c_{n-i} \in Q)[c_0 \dots c_{n-i} x_1 \dots x_i] \alpha [c_0 \dots c_{n-i} y_1 \dots y_j]. \quad (*)$$

Conversely, if β is a congruence on \mathbf{Q}^\wedge and $\alpha = \beta|_{\mathbf{Q}}$, then $\alpha^\wedge \subseteq \beta$, and $\alpha^\wedge = \beta \Leftrightarrow a_1 \dots a_i \beta b_1 \dots b_j \Rightarrow i = j$. \diamond

The fact that \mathbf{Q}^\wedge is a group allows us to describe α^\wedge by a corresponding normal subgroup H of \mathbf{Q}^\wedge . Namely:

3.2°. Let α be congruence on \mathbf{Q} , $H^\alpha = \{x \in Q^\wedge \mid x\alpha^\wedge e\}$.³⁾ Then $H^\alpha \subseteq Q^n$ is an invariant subgroup of \mathbf{Q}^\wedge , and $Q_{/\alpha} = \{xH^\alpha \mid x \in Q\}$.

Conversely, if K is an invariant subgroup of \mathbf{Q}^\wedge such that $K \subseteq Q^n$, then the corresponding congruence β^K on \mathbf{Q}^\wedge induces a congruence $\alpha = \beta^K|_{\mathbf{Q}}$ on \mathbf{Q} , such that $\alpha^\wedge = \beta^K$. \diamond

A more general statement of this property is the following: if L is an invariant subgroup of \mathbf{Q}^\wedge , then $K = L \cap Q$ is an invariant subgroup of \mathbf{Q}^\wedge , such that $B^L|_{\mathbf{Q}} = \beta^K|_{\mathbf{Q}}$, and in this case $(\beta^K|_{\mathbf{Q}})^\wedge \subseteq \beta^L$ whenever L is not a subset of Q^n .⁴⁾

4. Further on we will give a description of the congruences of an $n + 1$ -group \mathbf{Q} by corresponding non empty subsets of Q .

4.1°. Let \mathbf{Q} be an $n + 1$ -group, $a \in Q$ and let us define a binary operation $*$ in Q by:

$$x * y = [xa^{n-1}y].$$

Then $(Q; *)$ is a group isomorphic to \mathbf{Q}^n by the isomorphism φ :
 $x \mapsto a^{n-1}x$. \diamond

It is easy to prove that each congruence α on $(Q; [])$ is a congruence on $(Q; *)$ as well. Thus it induces an invariant subgroup $K = K^{a, \alpha}$ of $(Q; *)$. Consequently,

$$Q_{/\alpha} = Q_{/K} = \{x * K \mid x \in Q\} = \{[xa^{n-1}K] \mid x \in Q\}$$

and

$$[xa^{n-1}K] = [Ka^{n-1}x].$$

Thus, we can describe the α -classes by special subsets of Q , which depend on the choice of the element a .

Conversely, Let K be an invariant subgroup of $(Q; *)$. Then K induces an equivalence relation α on Q defined by $Q_{/\alpha} = Q_{/K}$. Now, the question

3) e is the neutral element of the group \mathbf{Q}^\wedge .

4) This is another way of stating Th. 2.7 of [5].

arises to find conditions such that α is a congruence on $(Q; [])$. We will answer this question considering the group Q^n , isomorphic to $(Q; *)$.

4.2°. α is a congruence on Q iff $a^{n-1}K$ is an invariant subgroup of Q^n , i.e. iff

$$(\forall x \in Q)xa^{n-1}K = a^{n-1}Kx. \quad \diamond$$

5. Let us recall that if H is a subgroup of an $n+1$ -group Q with the property

$$(\forall x \in Q)[xH^n] = [H^n x],$$

then we say that H is semiinvariant subgroup of Q ([2]). In this case H is a class of a congruence α on Q defined by

$$Q/\alpha = Q/H = \{[xH^n] \mid x \in Q\}.$$

The following property is also true:

The subgroup H of Q is semiinvariant in Q iff H^n is an invariant subgroup of Q^\wedge ([2]).

It is natural to ask the question whether every congruence on Q is obtained by a semiinvariant subgroup of Q . The answer of this question is negative. Let us give some examples to illustrate this fact.

Example 1. Let $Q = (2k+1)Z$ be the set of odd integers. We define a ternary operation by $[xyz] = x + y + z$, and obtain a 3-group, such that $Q^\wedge \cong (Z, +)$. Let α be the congruence on Q such that

$$Q/\alpha = \{\{4k+1 \mid k \in Z\}, \{4k+3 \mid k \in Z\}\}.$$

In this case there is no α -class that is a subgroup of Q .

Example 2. Let Q be the same as in Ex. 1, and α be the congruence on Q such that

$$Q/\alpha = \{\{6k+1 \mid k \in Z\}, \{6k+3 \mid k \in Z\}, \{6k+5 \mid k \in Z\}\}.$$

Then $\{6k+3 \mid k \in Z\}$ is the unique α -class that is a subgroup of Q .

Example 3. Let $Q = \{a_1, a_3, b_1, b_3\} \subseteq D_4$, and let define an operation $[]$ by $[xyz] = x \circ y \circ z$, where „ \circ ” is the operation in the dihedral group D_4 . Then Q is a 3-group, and $Q^\wedge = D_4$. If we define α to be the congruence on Q such that

$$Q/\alpha = \{\{a_1, a_3\}, \{b_1, b_3\}\},$$

then both the α -classes are subgroups of Q .

These examples initiate a problem of finding conditions under which an α -class is a subgroup of Q . The answer is given by the following property:

5.1°. Let Q be an $n+1$ -group, let α be a congruence on Q , $a \in Q$, and $H = H^{a,\alpha}$ be the non empty subset of Q induced by α and a (as in in 4.). Then: $[xa^{n-1}H]$ is a subgroup of Q iff $[x^{n+1}] \in [xa^{n+1}H]$, and in this case $[xa^{n-1}H]$ is a semiinvariant subgroup of Q . \diamond

6. In this last section of this paper we will give new proofs of some of the results of [5], using the universal covering group of an n -group Q .

Let α be a congruence on an $n+1$ -group Q . Then the α -equivalence class containing the element $z \in Q$ is called a z -ideal. Using the results in the previous section it is easily seen that a z -ideal I of an $n+1$ -group Q could be expressed in the form $I = zB$, where B is an invariant subgroup of

\mathbf{Q}^\wedge , such that $B \subseteq Q^n$. Using this fact and the properties of the universal covering group \mathbf{Q}^\wedge we can easily prove the following:

6.1°. (Th. 2.5 in [5].) If α and β are two congruences of an $n+1$ -group \mathbf{Q} , then

$$(\exists z \in Q) z^\alpha = z^\beta \Rightarrow \alpha = \beta. \quad \diamond$$

If I is a z -ideal, clearly $z \in I$ and $I = zB$ for some normal subgroup B of \mathbf{Q}^\wedge which is a subset of Q^n . Then, if $x, y \in I$, $x = zb_1$, $y = zb_2$, we have $[zb_1z^{1-n}z^{n-2}zb_2] = [zb_1b_2] \in zB = I$. If $x, y \in Q$ and $[x\bar{y}yy^{n-2}z] \in zB$, where \bar{y} is y^{1-n} , using the fact that $B \leq Q^\wedge$ we obtain that $[x\bar{x}x^{n-2}z] \in I = zB$ as well. Using the fact that $B \triangleleft Q^\wedge$ we easily check that if $y, x_1, \dots, x_n \in Q$, $[x_1^n z] \in zB$, then $[[yx_1^n]y^{-1}z] \in zB$, i.e. $[[yx_1^n]\bar{y}y^{n-2}z] \in zB$. Thus we have proven the „if” part of the following:

6.2°. (Th. 2.6 of [5]). Let \mathbf{Q} be an $n+1$ -group and $z \in Q$. The following two conditions are equivalent:

(i) I is z -ideal;

(ii) a) $z \in I$;

b) $x, y \in I \Rightarrow [x\bar{z}z^{n-2}y] \in I$;

c) $x, y \in Q$ and $[x\bar{y}y^{n-2}z] \in I \Rightarrow [y\bar{x}x^{n-2}z] \in I$;

d) $y, x_1, \dots, x_n \in Q$, $[x_1^n] \in I \Rightarrow [[yx_1^n]\bar{y}y^{n-2}z] \in I$.

To prove the converse we first show that if a)–d) hold, and if $B = z^{-1}I$, i.e. $I = zB$, then $B \subseteq Q^n$. Using a)–d) then we prove that $B \triangleleft Q^\wedge$. Namely, a) implies that the unity of \mathbf{Q}^\wedge is in B , b) and c) imply that B is a subgroup, and d) that B is an invariant subgroup of \mathbf{Q}^\wedge . \diamond

These two theorems give another characterization of a congruence of an $n+1$ -group \mathbf{Q} , i.e. each congruence α of an $n+1$ -group \mathbf{Q} is uniquely determined by its z -ideal.

References

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КОНГРУЕНЦИИ НА n -ГРУПИ

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Резиме

Во овој труд е даден опис на конгруенции на полиадични групи. Имено, ако $Q = (Q; [])$ е n -група, а Q^\wedge е нејзината универзална покривачка група, тогаш секоја конгруенција α на Q е окарактеризирана со нормална подгрупа $H_\alpha \subseteq \{a_1 \dots a_{n-1} \mid a_v \in Q\}$. Исто така, ако $a \in Q$ и $*$ е операција на Q дефинирана со $x * y = [xa^{n-2}y]$, тогаш $(Q; *)$ е група и секоја конгруенција α на Q е окарактеризирана со нормална подгрупа K од $(Q; *)$ таква што за секој $x \in Q$, $[xa^{n-2}K] = [a^{n-2}Kx]$. Покажано е и дека може да се најде n -група Q и конгруенција α на Q такви што виедна класа на еквивалентноста α да не биде n -подгрупа од Q , и најдени се нужни и доволни услови при кои класа на еквивалентноста α е n -подгрупа од Q .

На крајот од трудот дадени се нови, пократки докази на некои својства од [5] преку универзалната покривачка група.

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