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REDUCTION OF A BOUNDARY PROBLEM OF THE THIRD ORDER TO A BOUNDARY PROBLEM OF THE SECOND ORDER

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Abstract. The objective of this paper is a boundary problem of the third order whose solution is the second power of the solution of a boundary problem of the second order involving more general boundary conditions.

The paper [6] points out the possibility that a boundary problem of the higher order, under given conditions for the solution, could be reduced to a boundary problem of lower order. Thus, a boundary problem of the third order is reduced to a boundary problem of the second order under the Sturm's conditions of the first type. Also, using the same procedure, paper [1] considers a boundary problem of the fourth order and it's solution by reduction to a boundary problem of the second order, under the Sturm's conditions of the first type.

The boundary problems of the third order have not been treated very often because there are some difficulties involved in their solution.

1. Let us consider the homogeneous boundary problem of the third order which solution is the second power of the solution of the homogeneous boundary problem of the second order,

$$y'' + py' + qy = 0 \tag{1.1}$$

$$\alpha_{i0}y_a + \alpha_{i1}y'_a + \beta_{i0}y_b + \beta_{i1}y'_b = 0, \quad (i=1,2) \tag{1.2}$$

(where $y_c=y(c)$, $y'_c=y'(c)$ for $c=a,b$).

It is known [2] that the homogeneous boundary problem of the second order (1.1)-(1.2) has a solution if rank of the matrix of coefficients $\alpha_{\mu\nu}, \beta_{\mu\nu}$ $(\mu=1,2;\ \nu=0,1)$ is two.

The boundary problem with equation

$$y'' + \lambda y = 0, \qquad \lambda \geq 0$$

and boundary conditions (1.2) is solved in [4].

Let us define the solution of the boundary problem (1.1)-(1.2) taking the solution of the equation (1.1) in a form

$$y = e^{r_2 x}[C_2+C_1\int e^{(r_1-r_2)x}dx] \tag{1.3}$$

where r_1 and r_2 are roots of the characteristic equation

$$r^2 + pr + q = 0 \tag{1.4}$$

while C_1 and C_2 are arbitrary independent constants, [5].

Solution (1.3) of equation (1.1) should satisfy the boundary conditions (1.2) i.e.

$$\alpha_{i0}e^{r_2 a}(C_1 I_a+C_2)+\alpha_{i1}e^{r_2 a}[C_1(e^{(r_1-r_2)a}+r_2 I_a)+C_2 r_2] + \tag{1.5}$$

$$+ \beta_{i0}e^{r_2 b}(C_1 I_b+C_2)+\beta_{i1}e^{r_2 b}[C_1(e^{(r_1-r_2)b}+r_2 I_b)+C_2 r_2] = 0, \quad (i=1,2)$$

where $I_c = (\int e^{(r_1-r_2)x}dx)_{x=c}$.

The system (1.5) will have nontrivial solution of the constants C_1 and C_2 if its determinant is zero, i.e. if it applies

$$Ae^{r_1 a-r_2 b}+Be^{r_1 b-r_2 a}-(A_{10}+A_{11}r_2)e^{(r_1-r_2)a}+(A_{01}+r_2 A_{11})e^{(r_1-r_2)b}+$$

$$+(I_b-I_a)[r_2^2 A_{11}+r_2(A_{10}+A_{01})+A_{00}] = 0, \tag{1.6}$$

where

$$A = \begin{vmatrix} \alpha_{10} & \alpha_{11} \\ \alpha_{20} & \alpha_{21} \end{vmatrix} = \det(\alpha_{10},\alpha_{21}),$$

$$B = \begin{vmatrix} \beta_{10} & \beta_{11} \\ \beta_{20} & \beta_{21} \end{vmatrix} = \det(\beta_{10},\beta_{21}), \tag{1.7}$$

$$A_{ij} = \begin{vmatrix} \alpha_{1i} & \beta_{1j} \\ \alpha_{2i} & \beta_{2j} \end{vmatrix} = \det(\alpha_{1i},\beta_{2j}), \quad (i,j=0,1).$$

Equation (1.6) is a transcedental equation which can be used for definition of eigen values of the problem with eigen values (1.1)-(1.2) with known relation between p and q.

The number of non-complex roots of this equation is the number of the eigen values of the problem (1.1)-(1.2).

If $r_1=r_2$, the number of eigen values is at the most finite.
If $r_1 \neq r_2$, the number of eigen values is at the most infinite.

With precision to a multiplicative constant, the solution of the boundary problem (1.1)-(1.2) is as follows:

$$y_n = e^{r_2(x-a)}\{(\alpha_{10}+r_2\alpha_{11})(I_x-I_a)+(\beta_{10}+r_2\beta_{11})(I_x-I_b)e^{r_2(b-a)} - [\alpha_{11}e^{(r_1-r_2)a}+\beta_{11}e^{r_2(b-a)}e^{(r_1-r_2)b}]\}. \qquad (1.8)$$

For given relation between parameters p and q, the eigen functions of the problem with eigen values (1.1)-(1.2) become (1.8).

2. Let us define the boundary problem of the third order which solution is a second power of the boundary problem solution (1.1)-(1.2).

Putting

$$z = y^2 \qquad (2.1)$$

and, by differentiation, including the relation (1.1), it is obtained

$$z' = 2yy', \qquad (2.2)$$

$$z'' + pz' + 2qz = 2y'^2, \qquad (2.3)$$

$$z''' + 3pz'' + (2p^2+4q)z' + 4pqz = 0. \qquad (2.4)$$

The boundary conditions (1.2) will be as follows

$$\alpha_{io}y_a + \alpha_{i1}y'_a = -(\beta_{io}y_b+\beta_{i1}y'_b), \quad (i=1,2). \qquad (2.5)$$

By putting these two equations to the power of two, and multiplying them, in connection with (2.1), (2.2) and (2.3), the boundary conditions by z will be

$$\begin{aligned} 2\alpha_{io}\alpha_{jo}z_a + \sum_{k=o}^{1}\alpha_{ik}\alpha_{j\,k+1}z'_a + \alpha_{i1}\alpha_{j1}(z''_a+pz'_a+2qz_a) = \\ = 2\beta_{io}\beta_{jo}z_b + \sum_{k=o}^{1}\beta_{ik}\beta_{j\,k+1}z'_b+\beta_{i1}\beta_{j1}(z''_b+pz'_b+2qz_b), \\ (i,j)=\{(1,1),(1.2),(2.2)\}, \end{aligned} \qquad (2.6)$$

which together with equation (2.4) give the required boundary problem of the third order.

Accordingly, the boundary problem of the third order (2.4)-(2.6) has a solution

$$z_n = e^{2r_2(x-a)}\{(\alpha_{10}+r_2\alpha_{11})(I_x-I_a)+(\beta_{10}+r_2\beta_{11})(I_x-I_b)e^{r_2(b-a)} -$$
$$- [\alpha_{11}e^{(r_1-r_2)a}+\beta_{11}e^{r_2(b-a)}e^{(r_1-r_2)b}]\}^2. \tag{2.7}$$

In case of Sturm's boundary conditions

$$\begin{aligned} \alpha_{10}y_a + \alpha_{11}y'_a &= 0 \\ \alpha_{20}y_a + \alpha_{21}y'_a &= 0 \end{aligned} \tag{2.8}$$

the boundary conditions of the third order are easily provided

$$\begin{aligned} 2\alpha_{10}z_a + \alpha_{11}z'_a &= 0 \\ \alpha_{10}z'_a + \alpha_{11}(z''_a+pz'_a+2qz_a) &= 0 \\ 2\beta_{20}z_b + \beta_{21}z'_b &= 0 \\ \beta_{20}z'_b + \beta_{21}(z''_b+pz'_b+2qz_b) &= 0 \end{aligned} \tag{2.9}$$

These conditions define four triplets of boundary conditions which, with equation (2.4), define four boundary problems of the third order, the solution of which is

$$z_n = e^{2r_2(x-a)}[(\alpha_{10}+r_2\alpha_{11})(I_x-I_a)-\alpha_{11}e^{(r_1-r_2)a}]^2 \tag{2.10}$$

It is concluded from here that:

The four boundary problems of third order with the equation (2.4) and a triplet of boundary conditions (2.9) have a solution (2.10), which is a square of the solution of boundary problem of the second order (1.1)-(2.8).

<u>Example 1</u>. Let $p=q^2$. On the boundary problem with equation

$$y'' + py' + p^2y = 0 \tag{2.11}$$

and boundary conditions (2.8) are reduced the boundary problems of the third order with equation

$$z''' + 3pz'' + 6p^2z' + 4p^3z = 0 \tag{2.12}$$

and a triplet of the boundary conditions

$$\begin{aligned} 2\alpha_{10}z_a + \alpha_{11}z'_a &= 0 \\ \alpha_{10}z'_a + \alpha_{11}(z''_a+pz'_a+2p^2z_a) &= 0 \\ 2\beta_{20}z_b + \beta_{21}z'_b &= 0 \\ \beta_{20}z'_b + \beta_{21}(z''_b+pz'_b+2p^2z_b) &= 0 \end{aligned} \tag{2.13}$$

the eigen functions of which are as follows

$$z_n = e^{-px}\{(2\alpha_{10}-p\alpha_{11})\sin\frac{p\sqrt{3}(x-a)}{2} - \frac{p\sqrt{3}}{2}\cos\frac{p\sqrt{3}(x-a)}{2}\}^2 \quad (2.14)$$

where p is a root of the equation

$$tg\frac{p\sqrt{3}}{2}(b-a) = \frac{(\alpha_{11}\beta_{20}-\alpha_{10}\beta_{21})p\sqrt{3}}{2\alpha_{10}\beta_{20}-p(\alpha_{10}\beta_{21}-\alpha_{11}\beta_{20})+2p^2\alpha_{11}\beta_{21}}. \quad (2.15)$$

(The number of roots of (2.15) is the most infinite).

Example 2. Let $q = -\frac{3p^2}{4}$. To the boundary problem of the second order

$$y'' + py' - \frac{3p^2}{4}y = 0 \quad (2.16)$$

$$y(a) = y(b) = 0 \quad (2.17)$$

which has no solution, are reduced the boundary problems of the third order with the equation

$$z''' + 3pz'' - p^2z' - 3p^3z = 0 \quad (2.18)$$

and a triplet of boundary conditions

$$z_a = z'_a = z_b = z'_b = 0 \quad (2.19)$$

which also have no solution.

Example 3. Let p = 0. To the boundary problem of the second order with equation

$$y'' + qy = 0$$

and boundary conditions

$$y'(a) = y'(b) = 0 \quad (2.20)$$

are reduced the boundary problems of the third order with equation

$$z''' + 4qz' = 0$$

and a triplet of boundary conditions

$$z'_a = 2qz_a + z''_a = z'_b = 2qz_b + z''_b = 0$$

the eigen functions of which are as follows

$$z_n = \cos^2(x-a)\sqrt{q_n}$$

where q_n is a root of equation

$$\sin K = 0, \qquad (K=(b-a)\sqrt{q_n}),$$

and the eigen values are

$$q_n = \left(\frac{n\pi}{b-a}\right)^2, \qquad n=0,1,2,\ldots .$$

R E F E R E N C E S

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РЕДУКЦИЈА НА КОНТУРЕН ПРОБЛЕМ ОД ТРЕТ РЕД ВО КОНТУРЕН ПРОБЛЕМ ОД ВТОР РЕД

Слободанка С. Георгиевска

Р е з и м е

Во овој труд се разгледуваат контурни проблеми од трет ред чие решение е втора степен од решението на контурен проблем од втор ред со општи линеарни хомогени контурни услови. Ако контурните услови се Штурмови за контурниот проблем од втор ред, тогаш постојат четири контурни проблеми од трет ред кои имаат решение квадрат од решението на контурен проблем од втор ред.

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