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# SOME PROPERTIES OF $\mu$-APPROXIMATE $l_1$ SEQUENCIES IN BANACH SPACES

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## Abstract

The main result in this paper is the following: If $X$ is a Banach space, which contains an $\mu$-approximative $l_1$ system of vectors then there exists a subspace $L$ and a weighted shift operator $T : L \to L$ with a weighted sequence $(\lambda_i)$ such that $\inf_i(1-\epsilon_i)\cdot|\lambda_i|\,||x||-K \leq ||Tx|| \leq \sup_i|\lambda_i|\,||x||$, $K$-constant, for every $x$ in $L$ and for every decreasing sequence $0 < (\epsilon_i) < 1$ of real numbers, in case where $X = L_2[R]$ the norm of operator is estimate in whole space $X = L_2$.

## Introduction

Denote by $X$ the real or complex Banach space. Let $(x_i)_{i\in I}$ be a unit sequence in a real Banach space $X$ (where $I = \{1, 2, \ldots, n\}$ or $I = N$), and let $\mu > 0$. We say that $(x_i)$ is a $\mu$-approximate $l_1$ system if

$$\left\|\sum_{i\in A} \pm x_i\right\| \geq |A| - \mu$$

for all finite sets $A \subset I$ and for all choices of signs. Is valid the following

**Theorem 1.** [1] *Let $(x_i)_{i=1}^{n}$ be unit vectors in a real Banach space $X$ such that*

$$\left\|\sum_{i=1}^{n} \pm x_i\right\| = n$$

*for all choices of signs. Then*

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| = \sum_{i=1}^{n} |a_i|$$

*for all* $(a_i) \in l_1$.

**Theorem 2.** [2] *Suppose that* $(x_i)_{i=1}^{\infty}$ *is a* $\mu$*-approximate* $l_1$ *system. Then, given any decreasing sequence* $0 < (\epsilon_i) < 1$ *of positive numbers, there is a subsequence* $(y_i)$ *of* $(x_i)$ *such that*

$$\Big\|\sum_{i=1}^{\infty} a_i y_i\Big\| \geq \sum_{i=1}^{\infty} (1-\epsilon_i)|a_i|$$

*for all* $(a_i) \in l_1$. **Weighted shift operators in a Banach space with weight bounded sequence**

**Theorem 3.** *Let* $X$ *be a complex Banach space and let* $(x_i)$ *be a sequence of unit vectors. If*

$$||e^{i\xi_1} \cdot x_1 + \cdots + e^{i\xi_n} \cdot x_n|| = n, \forall \xi_1, \ldots, \xi_n \in [0, 2\pi]$$

*then the equality*

$$\Big\|\sum_{k=1}^{n} a_k x_k\Big\| = \sum_{k=1}^{n} |a_k|, a_k \in C, k = 1, \ldots, n$$

*holds true.*

**Proof.** Let $\xi_k = \arg a_k, k = 1, 2, \ldots, n$ and

$$x_0 = \sum_{k=1}^{n} e^{i\xi_k} \cdot x_k$$

Then by the Hanh-Banach Theorem there exists a functional $x^* \in X^*$, $||x^*|| = 1$ such that

$$x^*\Big(\sum_{k=1}^{n} e^{i\xi_k} \cdot x_k\Big) = \Big\|\sum_{k=1}^{n} e^{i\xi_k} \cdot x_k\Big\| = n$$

i.e

$$\sum_{k=1}^{n} e^{i\xi_k} x^*(x_k) = n\,. \tag{1}$$

Since $x_k$, $k = 1,2,\dots n$ are unit vectors and $\|x^*\| = 1$ we will have $|x^*(x_k)| \le 1$. Now, from the equality (1) we obtain $e^{i\xi_k} \cdot x^*(x_k) = 1$ and therefore $x^*(x_k) = e^{-i\xi_k}$. Further on,

$$\left|x^*(\sum_{k=1}^{n} a_k \cdot x_k)\right| \le \|x^*\| \cdot \left\|\sum_{k=1}^{n} a_k \cdot x_k\right\|$$

Hence,

$$\left|\sum_{k=1}^{n} a_k \cdot x^*(x_k)\right| \le \left\|\sum_{k=1}^{n} a_k \cdot x_k\right\|. \tag{2}$$

From the above inequality we will have

$$\left|\sum_{k=1}^{n} a_k x^*(x_k)\right| = \left|\sum_{k=1}^{n} |a_k| e^{i\arg a_k} e^{-i\arg a_k}\right| = \sum_{k=1}^{n} |a_k|. \tag{3}$$

Now, from (2) and (3) we will have

$$\sum_{k=1}^{n} |a_k| \le \left\|\sum_{k=1}^{n} a_k \cdot x_k\right\|.$$

On the other hand

$$\left\|\sum_{k=1}^{n} a_k x_k\right\| \le \sum_{k=1}^{n} \|a_k x_k\|$$

$$= \sum_{k=1}^{n} |a_k|.$$

and the proof of the Theorem is completed.

In the following, denote by $L$ the subspace of a real Banach space $X$, generated by the system of unit vectors $(e_i)_{i\in N}$, which satisfies the property

$$\left\|\sum_{i\in A} \pm e_i\right\| = k(A) \tag{4}$$

where $k(A)$ is the cardinal number of the set $A \subset I$ ($I = \{1,..,n\}$ or $I = N$), $N$-the set of natural numbers.

Define the weighted shift operator $T : L \to L$, by

$$Te_i = \lambda_i \cdot e_{i+1}$$

with bounded sequence $(\lambda_i)$. Then every vector $x \in L$ is of the form

$$x = \sum_{i \in N} a_i \cdot e_i$$

Further on, we will give an example for the subspace $L$ of the Banach space $X$.

**Example 1**. Let $L$ and $Y$ be the Banach spaces, where $L$ is isometricaly isomorphic to the Banach space $l_1$. Then,the direct sum $L \oplus Y$ is the Banach space.The basic vectors in $l_1$ are

$$e_1 = (1,0,0,\ldots)\ldots e_n = (0,0,\ldots,0,1,0,\ldots)\ldots$$

Now, $||e_i||_{l_1} = 1$ for every $i \in N$ and

$$\left\|\sum_{i=1}^{n} \pm e_i\right\|_{l_1} = ||(\pm1,\pm1,\ldots,\pm1,0,0,\ldots)||_{l_1}$$

$$= \sum_{i=1}^{n} |\pm 1| = n \equiv k(A)$$

where $A = \{1,2,\ldots,n\}, n \in N$.

For the set $L$ we will prove the following result.

**Lemma 1**. *The subset $L$ is a closed subspace of the Banach space $X$.*

**Proof**. Let $(x_n)$ be a Cauchy sequence and let $x_n = \sum\limits_{i=1}^{\infty} a_i{}^n e_i$. Then,

$$||x_m - x_n|| \to 0$$

Now from the Theorem 1 it follows that

$$||x_m - x_n|| = \sum_{i=1}^{\infty} |a_i{}^m - a_i{}^n| \to 0$$

Therefore, $(a_i{}^n)_{n \in N}$ is a Cauchy sequence of real numbers for every natural number $i$. Let $a_i{}^n \to a_i{}^0, i \to \infty$ and let $x_0 = \sum\limits_{i=1}^{\infty} a_i{}^0 e_i$. Obviosly, $x_0 = \lim\limits_{n \to \infty} x_n$ and $L$ is a closed subspace of the Banach space $X$.

**Theorem 4**. *Let $(X, ||.||)$ be a real Banach space and let $(e_i)_{i \in N}$ be a system of unit vectors, which satisfies condition (4). Then for every* *bounded sequence $(\lambda_i)$ there exists a closed subspace $L$ in $X$ and a weighted shift operator $T : L \to L$, defined as above, such that*

$$\inf_i (1-\epsilon_i)|\lambda_i| \cdot ||x|| - K \le ||Tx|| \le \sup_i |\lambda_i| \cdot ||x||$$

*for every $x \in L$, $K$ constant, and for every decreasing sequencies $0 < \epsilon_i < 1$.*

**Proof.** $(e_i)_{i\in N}$ is a 0-approximate system of vectors in $l_1$ (it follows from (4)).Let $(a_i) \in l_1$ from the Theorem 2 there is a subsequence $(y_i)$ of $(e_i)$ such that

$$\left\|\sum_{i=1}^{\infty} a_i \cdot y_i\right\| \ge \sum_{i=1}^{\infty} (1-\epsilon_i) \cdot |a_i| \tag{5}$$

for every sequence $(a_i) \in l_1$.

In the following we denote by $\{z_i : i \in N\} = \{e_i : i \in N\} \setminus \{y_i : i \in N\}$ and

$$\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} b_i y_i + \sum_{i\in I} c_i z_i$$

relation (5) is true for every $(a_i) \in l_1$, so it is true and for $(b_i)$(because $(b_i) \subset (a_i)$)

$$\left\|\sum_{i=1}^{\infty} b_i \cdot y_i\right\| \ge \sum_{i=1}^{\infty} (1-\epsilon_i) \cdot |b_i| \tag{6}$$

Now we estimate the relation

$$\left\|\sum_{i=1}^{\infty} a_i \cdot e_i\right\| = \left\|\sum_{i=1}^{\infty} b_i y_i + \sum_{i\in I} c_i z_i\right\|$$

$$\ge \sum_{i=1}^{\infty} (1-\epsilon_i)|b_i| - \left\|\sum_{i\in I} c_i z_i\right\|$$

$$\left\|\sum_{i\in I} c_i z_i\right\| \le \sum_{i\in I} |c_i| \; \| \, z_i \, \|$$

$$\le \sum_{i\in I} (1+\epsilon_i)|c_i| \Rightarrow -\left\|\sum_{i\in I} c_i z_i\right\|$$

$$\ge \sum_{i\in I} (1-\epsilon_i)|c_i| - 2\sum_{i\in I} |c_i|$$

and from that

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\| \geq \sum_{i=1}^{\infty}(1-\epsilon_i)|b_i| + \sum_{i\in I}(1-\epsilon_i)|c_i|$$

$$-2\sum_{i\in I}|c_i| = \sum_{i=1}^{\infty}(1-\epsilon_i)|a_i| - K\,.$$

Further on, denote by $L$ the subspace generated by vectors $(e_i)_{i\in N}$ and let $T : L \to L$, be shifted operator with bounded sequence $(\lambda_i)$.For every vector $x = \sum_{i\in N} a_i e_i \in L$, it follows that

$$\begin{aligned}||Tx|| = \left\|\sum_{i\in N}\lambda_i \cdot a_i \cdot e_{i+1}\right\| &\geq \sum_{i\in N}(1-\epsilon_i)|\lambda_i|\cdot|a_i| - K\\ &\geq \inf_i (1-\epsilon_i)|\lambda_i|\sum_{i\in N}|a_i| - K\\ &= \inf_i (1-\epsilon_i)|\lambda_i|\cdot||x|| - K\end{aligned}$$

On the other hand

$$\begin{aligned}||Tx|| &= \left\|\sum_{i\in N}\lambda_i \cdot a_i \cdot e_{i+1}\right\|\\ &\leq \sum_{i\in N}|\lambda_i||a_i|||e_{i+1}||\\ &= \sum_{i\in N}|\lambda_i||a_i| \leq \sup_i |\lambda_i|\cdot||x||\end{aligned}$$

and the proof of the Theorem is completed.

Further on, consider a Hilbert space $X = L_2(R)$,with respect to the inner product

$$(x,y) = \int_R x(t)\overline{y(t)}dt$$

which is spanned by a sequence of Hermite functions

$$\begin{aligned}h_n(x) &= \frac{2^{\frac{1}{4}}}{\beta_n}g_n(x), g_n(x)\\ &= (-1)^n\cdot e^{\pi\cdot x^2}\cdot\frac{d^n}{dx^n}(e^{-2\pi x^2}),\quad \beta_n = ((4\pi)^n\cdot n!)^{\frac{1}{2}}\end{aligned}$$

**Lemma 2.** [5] *The sequencies of hermite functions hold the following properties*

(1) *it is a orthonormal in* $L_2(R)$

(2) *it is a basic in* $L_2(R)$

*Now, there is easy to see that*

$$\left\|\sum_{i\in A} h_i\right\| = \left\|\sum_{i\in A} \sqrt{(h_i,h_i)}\right\| = k(A)$$

*for every finite subset* $A$ *of natural numbers* $N$.*It means that* $(h_i)$ *is a 0-approximate* $l_1$ *system of unit vectors in* $L_2(R)$.

At the end we obtain the following result,which is the direct consequence of Theorem 4.

**Corollary 1.** *Let* $L_2(R)$ *be a Hilbert space as described above and let* $(h_i)$ *be a system of Hermite functions, and* $T$ *operator defined in* $L_2(R)$ *with* $T(h_i) = \lambda_i h_{i+1}$,*the norm of this operator hold the relation*

$$\inf_i (1-\epsilon_i)|\lambda_i| \cdot ||x|| - K \leq ||Tx|| \leq \sup_i |\lambda_i| \cdot ||x||$$

*for every* $x \in L_2(R)$, $K$ *constant, and for every decreasing sequencies* $0 < (\epsilon_i) < 1$.

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# НЕКОИ ОСОБИНИ НА $\mu$-АПРОКСИМИРАЧКИ $l_1$ НИЗИ ВО БАНАХОВ ПРОСТОР

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**Р е з и м е**

Главниот резултат во оваа работа е следниот: Ако $X$ е Банахов простор, којшто содржи $\mu$-апроксимативен $l_1$ систем од вектори, тогаш постои потпростор $L$ и тежински шифт оператор $T : L \to L$ со тежинска низа $(\lambda_i)$ така што $\inf_i(1-\epsilon_i)\cdot|\lambda_i|\,\|x\| - K \leq \|Tx\| \leq \sup_i|\lambda_i|\,\|x\|$, $K$-константа, за секое $x$ во $L$ и за секоја опаѓачка низа $0 < (\epsilon_i) < 1$ на реални броеви, во случај каде $X = L_2[R]$ нормата на операторот е оценета во просторот $X = L_2$.

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