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# TWO ALGORITHMS FOR FAREY TREE

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## Abstract

The Farey tree is a binary tree containing all rational numbers from [0, 1] in an ordered way. It is constructed hierarchically, level by level, using the Farey mediant sum. Some known algorithms, connected with the Farey tree construction are reviewed and two new algorithms are proposed.

## 1. Introduction

One of almost every day used mathematical theorems claims that the set **Q** of rational numbers is countable, i.e., they may be "ordered" as a sequence $r_1, r_2, r_3, \ldots, r_n, \ldots$ There are infinitely many ways to construct this sequence. One of these, known as *Farey tree*, has important applications in the theory of coupled oscillators and the so called "golden route to chaos" ([1], [3], [4], [5]).

The Farey tree is a collection of sets (called levels) $FT = \{T_{-1}, T_0, T_1, \ldots\}$, where $T_{-1} = \{r_{-1} = 1/1, r_0 = 0/1\}$ is called *seed of the tree*. The $n$-th level $T_n = \{r_{2^n}, \ldots, r_{2^{n+1}-1}\}$, $n = 0, 1, 2, \ldots$, is the decreasing sequence of rationals $r_j \in (0, 1)$. The 0-th level, $T_0 = \{r_1 = 1/2\}$ is the *root* of $FT$. Further levels are $T_1 = \{r_2 = 2/3,\ r_3 = 1/3\}$, $T_2 = \{r_4 = 3/4,\ r_5 = 3/5,\ r_6 = 2/5,\ r_7 = 1/4\}, \ldots$. One can identify $FT$ with the infinite binary graph, shown in Figure 1.

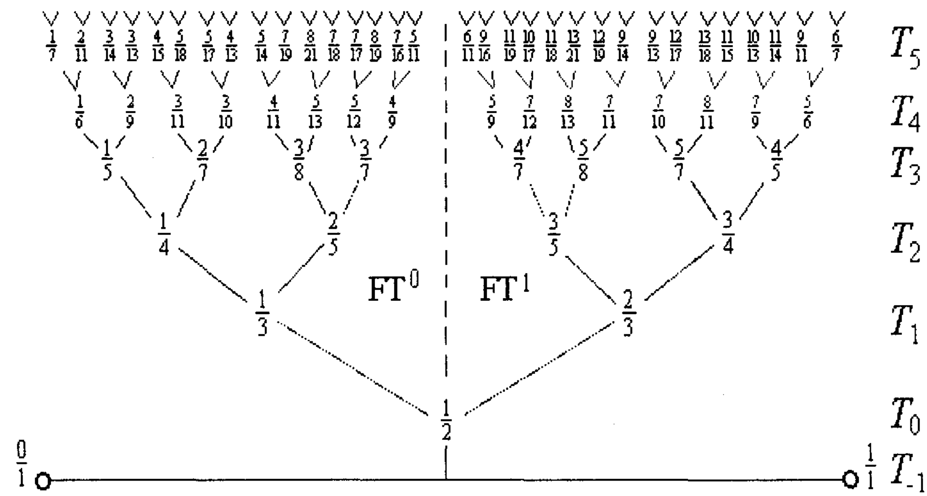


Figure 1. Farey tree

It is known that the set of vertices of $FT$ is isomorphic with the set $\mathbf{Q}[0,1]$ of rationals from the segment [0, 1]. The hierarchic construction of the Farey tree is closely related to binary operation $\oplus$, called *Farey sum*, defined on $\mathbf{Q}[0,1]$ by $p/q \oplus r/s = (p+r)/(q+s)$. The result, $(p+r)/(q+s)$ is called *mediant* of $p/q$ and $r/s$, due to the fact that $p/q \leq (p+r)/(q+s) \leq r/s$. Note that $\{\mathbf{Q}[0,1], \oplus\}$ is commutative semi-group.

**Lemma 1.** ([4]) *Every rational* $\rho \in (0,1)$ *can be uniquely expressed as a mediant of two distinctive rationals* $\rho_1, \rho_2 \in (0,1)$, *i.e.,* $\rho = \rho_1 \oplus \rho_2$, and *occurs uniquely as a vertex of the Farey tree.*

Two rationals $p/q$ and $r/s$ are called *adjacents* (or *Farey neighbors*) if $|ps - qr| = 1$. Note that the binary relation of adjacency is non-reflexive and non-transitive but symmetric.

**Lemma 2.** If $\rho_1, \rho_2 \in \mathbf{Q}[0,1]$ are adjacent rationals, then $\rho = \rho_1 \oplus \rho_2$ is adjacent to $\rho_1$ and $\rho_2$.

**Proof.** Let $\rho = p/q$, $\rho_1 = p_1/q_1$ and $\rho_2 = p_2/q_2$. Since $p_1/q_1$ and $p_2/q_2$ are adjacents, $|p_1q_2 - p_2q_1| = 1$. By definition, $p = p_1 + p_2$ and $q = q_1 + q_2$, and $|pq_1 - p_1q| = |(p_1+p_2)q_1 - p_1(q_1+q_2)| = |p_2q_1 - p_1q_2| = 1$. Similarly, $|pq_2 - p_2q| = |(p_1+p_2)q_2 - p_2(q_1+q_2)| = |p_1q_2 - p_2q_1| = 1$. □

The Farey tree introduces order in the set [0, 1], i.e. it embodies one of the possible mappings $\mathbf{N} \to \mathbf{Q}$ . This mapping is given by

**Theorem 1** [5]. Let $(b_0, b_1, \ldots, b_m)$, $b_i \in \{0,1\}$, $b_0 = 1$ be the sequence of binary digits making the binary expansion of $n \in \mathbf{N}$. Let the sequence $(a_0, a_1, \ldots, a_k)a_j \in \mathbf{N}$, represents cardinal numbers of subsets of successive units or zeros in the sequence $(b_0, b_1, \ldots, b_{m-1}, b_m, b_m)$. Then,

$$n \mapsto r_n = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}} = [a_0, a_1, \ldots, a_k].$$

**Remark 1.** Note that $a_k$ can not be less than 2 due to duplication of the last binary digit. In fact, there is an ambiguity in continued fraction expansion since for $a_k \geq 2$, the identity holds

$$[a_0, a_1, \ldots, a_k] = a_0, a_1, \ldots, a_k - 1, 1].$$

In virtue of Theorem 1, the rationals are ordered as follows: $r_1 = 1/2$, $r_2 = 2/3$, $r_3 = 1/3$, $r_4 = 3/4$, $r_5 = 3/5$, $r_6 = 2/5$, $r_7 = 1/4$, $r_8 = 4/5$, $r_9 = 5/7$, $r_{10} = 5/8$, $r_{11} = 4/7$, $r_{12} = 3/7$, $r_{13} = 3/8$, $r_{14} = 2/7$, $r_{15} = 1/5$, $r_{16} = 5/6$, $r_{17} = 7/9$ etc. Also, note that $r_k$ belongs to the level $T_{\lfloor \log_2 k \rfloor}$ of the Farey tree, where $\lfloor x \rfloor$ represents the entire part of $x$. Any element $r_k \in T_n$ $(k = 1, 2, \ldots)$ produces two "children", the left one $r_{2k+1}$, and the right one $r_{2k}$, or symbolically $r_k \mapsto (r_{2k+1}, r_{2k})$. It is easy to see that both belong to the next level $T_{n+1}$, since $\lfloor \log_2(2k) \rfloor = \lfloor \log_2(2k+1) \rfloor = n+1$. Also, each $r_k \in T_n$ $(k = 2, 3, \ldots)$, has two "parents". If $k$ is even, then $r_{k/2} \in T_{n-1}$ is the left one whilst the right one is $r_\mu \in T_\nu$ $(\nu < \lfloor \log_2(k-1) \rfloor)$. If $k$ is odd, then $r_{(k-1)/2} \in T_{n-1}$ is the right "child" and $r_\mu \in T_\nu$, $(\nu < \lfloor \log_2(k-1) \rfloor)$.

In [1] Cvitanović gave the following formal definition of the Farey tree level:

**Definition 2.** The $n$-th Farey tree level $T_n$ is the monotonically increasing sequence of $2^n$ contained fractions $[a_0, a_1, \ldots, a_k]$ whose entries $a_i \geq 1$, $i = 1, 2, \ldots, k-1$, $a_k \geq 2$, add up to $n+2$.

For example, $T_2 = \{[4], [2, 2], [1, 1, 2], [1, 3]\} = \{1/4, 2/5, 3/5, 3/4\}$.

Let $\alpha$ and $\beta$ represent the following simple mappings of rationals from $(0, 1)$:

$$\alpha\colon [a_0, a_1, \ldots, a_n] \mapsto [a_0, a_1, \ldots, a_n - 1, 2],$$

$$\beta\colon [a_0, a_1, \ldots, a_n] \mapsto [a_0, a_1, \ldots, a_n + 1].$$

One can associate the next algorithm to this definition:

*Algorithm* 1. ([1]). The "children" of the Farey tree element $r_k$ $(k = 1, 2, \ldots)$ are

$$r_{2k+1} = \alpha(r_k), r_{2k} = \beta(r_k), \quad \text{if } k \text{ is even;}$$

$$r_{2k+1} = \beta(r_k), r_{2k} = \alpha(r_k), \quad \text{if } k \text{ is odd.}$$

**Example 1.** The Algorithm 1 gives: $1/2 = [2] \mapsto \{[3], [1,2]\}$, $1/3 = [3] \mapsto \{[4], [2,2]\}$, $2/3 = [1,2] \mapsto \{[1,1,2], [1,3]\}$, $1/4 = [4] \mapsto \{[5], [3,2]\}$, $2/5 = [2,2] \mapsto \{[2,1,2], [2,3]\}$, $3/5 = [1,1,2] \mapsto \{[1,1,3], [1,1,1,2]\}$ , $3/4 = [1,3] \mapsto \{[1,2,2], [1,4]\}, \ldots$ etc.

Based on Theorem 1, Kappraff and Adamson [2] gave an effective algorithm for constructing the Farey tree. Here, this algorithm is formally encoded and an inverse algorithm is given as well.

*Algorithm* 2. (Direct algorithm, $n \mapsto r_n$ ($\mathbf{N} \to \mathbf{Q}$)). Let $n \in \mathbf{N}$, $n = (b_0 b_1 \ldots b_m)_2 \Rightarrow (b_0 b_1 \ldots b_m b_m)_2 \Rightarrow (\beta_0, \beta_1, \ldots, \beta\mu)$, where $\beta_j = \text{card}\{b_{\nu+1}, b_{\nu+2}, \ldots, b_{\nu+j}\}$, $b_l + b_{l+1} = 1 \wedge b_l b_{l+1} = 0$. Then,

$$r_n = [\beta_0, \beta_1, \ldots \beta_\mu] .$$

**Example 2.** Algorithm 2 gives $r_1 = 1/2$, $r_2 = 2/3$, $r_3 = 1/3$, $r_4 = 3/4$, $r_5 = 3/5$, $r_6 = 2/5$, $r_7 = 1/4$, $r_8 = 4/5$, $r_9 = 5/7$, $r_{10} = 5/8$, $r_{11} = 4/7, r_{12} = 3/7, r_{13} = 3/8, r_{14} = 2/7$, $r_{15} = 1/5$, $r_{16} = 5/6$, $r_{17} = 7/9$ etc.

The following inverse algorithm gives the position of a given rational number (from [0, 1]) in the hierarchy of the Farey tree.

*Algorithm* 3. (Inverse algorithm, $r = p/q \mapsto n$ ($\mathbf{Q} \to \mathbf{N}$)). Let $r \in \mathbf{Q}$ with continued fraction representation $r = [\beta_0, \beta_1, \ldots \beta\mu]$. Then, in binary representation, $n = (1\ldots1)(0\ldots0)(1\ldots1)\ldots(1\ldots1)$ blocks of "ones" and "zeros" contain $\beta_0, \beta_1, \ldots, \beta_{\mu-1}$ and $\beta_\mu - 1$ elements, respectively.

**Example 3.** The application of Algorithm 3 reveals that 6/7 is the 32-nd member of the Farey tree hierarchy, while 1/7 is the 63–rd. In fact, these two rationals are boundary elements of the fifth level of the Farey tree.

There is another algorithm ([5]) that allows calculating of two immediate successors of element $r_n$ of the Farey tree, and these are $r_{2n}$ and $r_{2n+1}$. For ex. the immediate successors of $r_9 = 5/7$ are $r_{18} = 8/11$ and $r_{19} = 7/10$. The algorithm is based on the identity mentioned in Remark 1.

*Algorithm* 4. (*Immediate successors*) Let $r_n$, $n \geq 1$, be any element of the Farey tree with the continued fraction expansion $r_n = [\beta_0, \beta_1, \ldots, \beta_\mu]$. Then,

$$r_{2n} = [\beta_0, \beta_1, \ldots, \beta_\mu + 1], \quad r_{2n+1} = [\beta_0, \beta_1, \ldots, \beta_\mu - 1, 2] \quad \text{for even n;}$$

$$r_{2n} = [\beta_0, \beta_1, \ldots, \beta_\mu - 1, 2], \quad r_{2n+1} = [\beta_0, \beta_1, \ldots, \beta_\mu + 1] \quad \text{for odd} \quad n.$$

**Example 4.** The immediate successors of $r_{12} = [2,3] = 3/7$ are $r_{24} = [2,4] = 4/9$ and $r_{25} = [2,2,2] = 5/12$; Successors of $r_{23} = [1,1,4] = 5/9$ are $r_{46} = [1,1,3,2] = 9/16$ and $r_{47} = [1,1,5] = 6/11$.

## 2. New algorithms

The Farey tree is *complementary symmetric* in the sense that each level $T_n (n = 1, 2, \ldots)$ is invariant under automorphism $\varphi : r \mapsto 1 - r$. In other words, the equality $\varphi(r_{2^n+k}) = 1 - r_{2^{n+1}-k-1}$ holds for $k = 0, 1, \ldots, 2^n$. If $T_n = \{r_{2^n}, \ldots r_{2^{n+1}-1}\}$ is taken as an ordered $2^n$-tuple, then $T_n$ is a (strictly) decreasing and accordingly $\varphi(T_n)$ is a (strictly) increasing sequence. The vertical axis, passing through the root $T_0 = \{1/2\}$ splits the tree into left sub-tree $FT^0$ and right sub-tree $FT^1$ (see Fig. 1). The left one $FT^0$, contains rationals from $(0, 1/2)$ and will be called "0-subtree". Elements of the right sub-tree $FT^1$ fall into the complementary subinterval $(1/2, 1)$, so this sub-tree may be called "1-subtree". This means that each level $T_n$ splits into two sub-levels: "0-level" $T_n^0$ and "1-level" $T_n^1$. In fact,

$$T_n^1 = (r_{2^n}, \ldots, r_{3 \cdot 2^{n-1}-1}), \quad T_n^0 = (r_{3 \cdot 2^{n-1}}, \ldots, r_{2^{n+1}-1}), \tag{3}$$

are decreasing sequences. For ex. $T_3^1 = (4/5, 5/7, 5/8, 4/7)$, $T_3^0 = (3/7, 3/8, 2/7, 1/5)$.

**Theorem 2.** The elements of the "1-subtree" are represented by $[1, a_1, a_2, \ldots, a_k]$ ] and the elements of the "0-subtree" are represented by $[a_0, a_1, \ldots, a_k]$, $a_0 \geq 2$.

**Proof.** Let $p/q \in T_n^1$ $(n = 1, 2, \ldots)$. Then, $1/2 < p/q < 1$, so $1 < q/p < 2$ and

$$\frac{p}{q} = \frac{1}{\frac{q}{p}} = \frac{1}{1 + \frac{q-p}{p}} = \frac{1}{1 + \frac{1}{Q}} \tag{4}$$

where $Q = \frac{p}{q-p} > 1$ is the consequence of $0 < (q-p)/p < 1$ that follows from the supposition $1/2 < p/q < 1$. Assuming that $1/Q = [a_1, a_2 \ldots, a_k]$ gives $p/q = [1, a_1, a_2 \ldots, a_k]$.

On the other hand, if $p/q \in T_n^0$ $(n = 1, 2, \ldots)$, then $1 - p/q \in T_n^1$ and therefore, by (4),

$$1 - \frac{p}{q} = 1 - \frac{1}{1 + \frac{1}{Q}} = \frac{1}{1 + Q} = \frac{1}{a_0 + \frac{1}{\rho}},$$

where $a_0 = 1 + \lfloor Q \rfloor > 2$ and $0 < \rho = Q - \lfloor Q \rfloor < 1$. If $\rho$ has continuous fraction expansion $[a_1 \ldots, a_k]$, then $1 - p/q = [a_0, a_1, \ldots, a_k]$, $a_0 \geq 2$, which completes the proof. □

Let the following three transformations of rationals from $\mathbf{Q}(0,1)$ be introduced:

$$A\colon [a_0, a_1, \ldots, a_k] \mapsto [a_0+1, a_1, \ldots a_k];$$
$$B\colon [a_0, a_1, \ldots, a_k] \mapsto [1, a_0, a_1, \ldots a_k]; \tag{5}$$
$$C\colon [a_0, a_1, \ldots, a_k] \mapsto [1, a_0-1, a_1, \ldots a_k]\,.$$

**Corollary 1.** $C(T_n^0) = T_n^1$, i.e., $C(r_{2^{n+1}-k}) = r_{2^n+k-1}$, $k = 1, 2, \ldots, 2^{n-1}$. Inversely, $C^{-1}(T_n^1) = T_n^0$.

**Proof.** Since $r_{2^n+k-1} \in T_n^0$ $(k = 1, 2, \ldots, 2^{n-1})$, by Th. 2, $r^n_{2+k-1} = [a_0+1, a_1, \ldots, a_k]$, $a_0 \geq 1$. Now, $C(r_{2^{n+1}-k}) = C([a_0+1, a_1, \ldots, a_k]) = [1, a_0, a_1, \ldots, a_k]$. It is easy to see that

$$[1, a_0, a_1, \ldots, a_k] = \frac{1}{1 + [a_0, a_1, \ldots, a_k]} \quad \text{and}$$

$$[a_0+1, a_1, \ldots, a_k] = \frac{1}{1 + \dfrac{1}{[a_0, a_1, \ldots, a_k]}},$$

which yields $[1, a_0, a_1, \ldots, a_k] = 1 - [a_0+1, a_1, \ldots, a_k] = r_{2^n+k-1}$. □

**Remark 2.** It follows from Corollary 1 that $FT^1 = C(FT^0)$ and $FT^0 = C^{-1}(FT^0)$.

**Lemma 3.** The mapping A maps $T_{n-1}$ onto $T_n^0$.

**Proof.** First, note that $\operatorname{card}(T_{n-1}) = \operatorname{card}(T_n^0) = 2^{n-1}$, $n = 1, 2, \ldots$ Suppose that $\rho \in T_{n-1}$. Then, by Definition 2, $\rho = [a_0, a_1, \ldots, a_k]$, where $a_i \geq 1$, $i = 1, 2, \ldots, k-1$, $a_k \geq 2$, and $\Sigma a_i = n-3$. Since $A(\rho) = A([a_0, a_1, \ldots, a_k]) = [a_0+1, a_1, \ldots, a_k]$, then the sum of partial quotients is $1 + \Sigma a_i = n-2$, so $A(\rho) \in T_n$. Further, $a_0 \geq 2$ which, by Theorem 2 guarantees that $A(\rho)$ belongs to the "0-subtree". This gives $A(\rho) \in T_n^0$. In addition, $A$ is "onto", so $A(T_{n-1}) = T_n^0$. □

**Lemma 4.** The mapping B maps $T_{n-1}$ onto $\searrow T_n^1$. Here, the sign "$\searrow$" denotes a decreasing reordering of a sequence.

**Proof.** Similarly as in the previous proof, $\operatorname{card}(T_{n-1}) = \operatorname{card}(T_n^1)$ which makes $B$ "onto". For any $\rho = [a_0, a_1, \ldots, a_k] \in T_{n-1}$, $a_i \geq 1$, $i = 1, 2, \ldots, k-1$, $a_k \geq 2$, and $\Sigma a_i = n-3$ (Definition 2). Also, $B(\rho) = [1, a_0, a_1, \ldots, a_k]$, with the sum of partial quotients $1 + \Sigma a_i = n-2$. Therefore, $B(\rho) \in T_n$. Further, the first partial quotient of $B(\rho)$ is 1 making it the member of the "1-subtree" or $B(\rho) \in T_n^1$. □

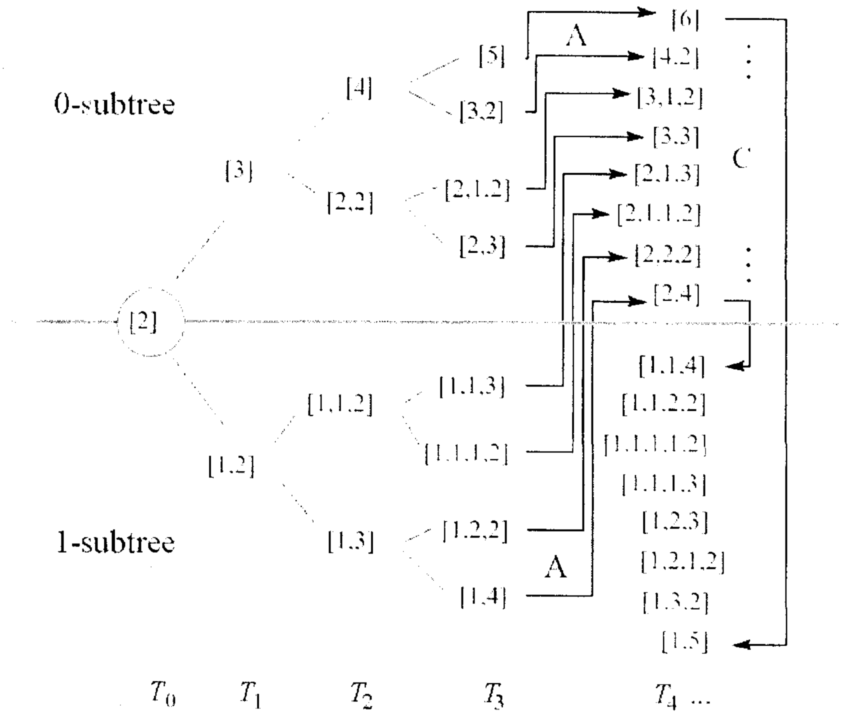


Figure 2. Algorithm A

Now one has the following theorem.

**Theorem 3.** The Farey tree may be generated by each of the next two procedures:

*Algorithm A:*

$A(r_{2^{n-1}+k}) = r_{3\cdot 2^{n-1}+k}, C(r_{3\cdot 2^{n-1}+k}) = r_{3\cdot 2^{n-1}-k-1}, \; k = 0,1,\ldots,2^{n-1}$, $n \in \mathbf{N}$.

*Algorithm B:*

$B(r_{2^{n-1}+k}) = r_{3\cdot 2^{n-1}-k-1}, \; C^{-1}(r_{3\cdot 2^{n-1}-k-1}) = r_{3\cdot 2^{n-1}+k}$, $k = 0,1,\ldots,2^{n-1}-1, \; n \in \mathbf{N}$.

**Proof.** The proof directly follows from Lemma 3 and 4 and Corollary 1. □

**Remark 3.** The Algorithms A and B are illustrated in Figures 2 and 3.

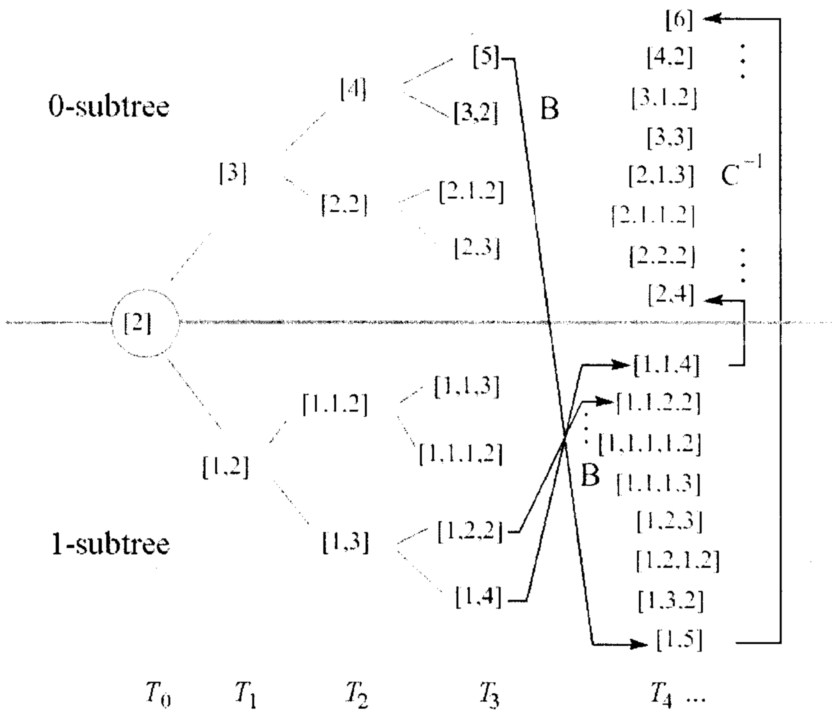


Figure 3. Algorithm B

**Remark 4.** There is one and only one sequence of successive descendants of the Farey tree that is generated by applying the B transformation only (starting by $1/1 = [1]$). In "ending one" notation this sequence is

$$[1], [1,1], [1,1,1], [1,1,1,1], [1,1,1,1,1], [1,1,1,1,1,1], \ldots \tag{6}$$

which is but the sequence of Fibonacci numbers ratios $F_i/F_{i+1}$, for $i \in \mathbf{N}$,

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \qquad \square$$

where $F_0 = 0$, $F_1 = 1$, $F_{i+1} = F_i + F_{i-1}$, $i \in \mathbf{N}$. The leftmost and rightmost quotients in the basis of the tree are $0/1 = F_0/F_1$ and $1/1 = F_1/F_2$. The limit of (6) is the Golden ratio $\gamma = (\sqrt{5} - 1)/2 \approx 0.61803398874989484 82$. By introducing the sequence $\Phi_i = Fi/F_{i+1}$, $i \in \mathbf{N}$, one can see that transformation $B$, given by (5) shifts this sequence, i.e. $B(\Phi_i) = \Phi_{i+1}$. Or, the sequence $\Phi = \{\Phi_i | i \in \mathbf{N}\}$, is invariant under $B$, since $B(\Phi) \subseteq \Phi$.

The sequence (6) belongs to the "1-subtree" or $FT^1$. Its complement is the sequence "symmetric" to (6)

$$[0], [2], [2,1], [2,1,1], [2,1,1,1], [2,1,1,1,1], \ldots \tag{7}$$

or $0/1, 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34\ldots$, which is the sequence $\{F_{i-1}/F_{i+1},\ i \in \mathbf{N}\}$. This time, the limit is $\gamma^2 = C(\gamma) = 1 - \gamma = (3 - \sqrt{5})/2 \approx 0.381966011250105 1518$.

Both sequences (6) and (7) play an important role in the Chaos theory. They are known as typical *quasiperiodic routes to chaos.* In fact, if a dynamic system contains two periodic oscillators with different frequencies, $f_1$ and $f_2$ ($f_1 < f_2$), the regime in the system tries to preserve the state where the ratio $f_1/f_2$ is the simplest rational number, say 1. Then, $f_1 : f_2 = 1:1$ which is called *optimal resonance* or 1:1 *mode-locking* regime. If this is not possible, the system "jumps" to the "reserve" mode-locking state, $f_1/f_2 = 1/2$. If, by some reason, this state is not possible, the system passes to the next "the simplest" mode-locking possibility, $f_1/f_2 = 2/3$ (or $f_1/f_2 = 1/3$), and so on, up the Farey tree. If, the system follows the sequence of mode-locking Fibonacci ratios given by (6), this is called golden route to chaos, and this route is the quickest one.

## 3. Conclusion

In this paper, two new algorithms for Farey tree construction are given. It is based on three simple transformations of continued fractions (5), and the complementary symmetry of the Farey tree. The most important subsequence of the Farey tree is the "golden one", $\{F_i/F_{i+1},\ i \in \mathbf{N}\}$ where $\{F_i\}$ is the well-known Fibonacci sequence. In the same time, it is the only sequence taken along the Farey tree which is invariant under the mapping B.

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# ДВА АЛГОРИТМА ЗА ФАРИЕВОТО ДРВО

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**Резиме**

Фариевото дрво е бинарно дрво кое ги содржи сите рационални броеви од $[0,1]$ подредени на посебен начин. Тоа се конструира хиерархиски, ниво по ниво, со користење на средна Фариева сума. Прикажани се некои познати алгоритми поврзани за конструкција на Фариевото дрво и предложени се два нови алгоритма.

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