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A NOTE ON THE INVERSE THEOREM FOR A GENERALISED MODULUS OF SMOOTHNESS

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Abstract

We prove the theorem inverse to Jackson's theorem for a modulus of smoothness of the first order generalised by means of asymmetric operator of generalised translation.

Introduction

The relation between the modulus of smoothness and the best approximation by trigonometric polynomials od 2π-periodic function is well-known. In the case od non-periodic functions there is no such relation between their moduli of smoothness and the bestapproximations by algebraic polynomials. An analogy with the 2π-periodic case takes place of the ordinary modulus of smoothness is replaced by a generalised modulus of smoothness (see e.g. [4]-[6]). In number of papers generalised moduli of smoothness are introduced by neabs of the generalised symmetric operator of translation ([2], [3], [6]).

In [1], an asymmetric operator of generalised translation is intoduced, by means of it the generalised modulus of smoothness of the first order is defined, and the theorem of coincidence of the class of functions defined by that modulus with the class of functions with given order of the best approximation by algebraic polynomials is proved. In the present paper we prove the theorem inverse to the Jackon's theorem related to that modulus of smoothness.

1. Definitions.

By L_p we denote the set of functions f measurable on the segment $[-1,1]$ such that for $1 \le p < \infty$

$$\|f\|_p = \left(\int_{1-}^{1} |f(x)|^p dx \right)^{1/p} < \infty ,$$

and for $p = \infty$

$$\|f\|_\infty = \text{ess} \sup_{-1 \le x \le 1} |f(x)| < \infty .$$

Denote by $L_{p,\alpha}$ the set of functions f such that $f(x)(1-x^2)^\alpha \in L_p$, and put

$$\|f\|_{p,\alpha} = \|f(x)(1-x^2)^\alpha\|_p .$$

By $E_n(f)_{p,\alpha}$ we denote the best approximation of the function $f \in L_{p,\alpha}$ by algebraic polynomials of degree not greater than $n-1$, in $L_{p,\alpha}$ metrics, i.e.

$$E_n(f)_{p,\alpha} = \inf_{P_n} \|f - P_n\|_{p,\alpha} ,$$

where P_n are algebraic polynomials of degree not greater than $n-1$.

For a function f we define the operator of generalised transformation $\hat{T}(f,x)$ by

$$\hat{T}(f,x) = \frac{1}{\pi(1-x^2)} \int_0^\pi \Big(-\Big(x \cos t + \sqrt{1-x^2} \sin t \cos\varphi \Big)^2 - 2\sin^2 t \sin^2\varphi +$$

$$+ 4(1-x^2)\sin^2 t \sin^4\varphi \Big) f\Big(x \cos t + \sqrt{1-x^2} \sin t \cos\varphi \Big) d\varphi .$$

By means of that operator of generalised transformation we define the generalised modulus of smoothness by

$$\hat{\omega}(f,\delta)_{p,\alpha} = \sup_{|t| \le \delta} \|\hat{T}_t(f,x) - f(x)\|_{p\alpha} .$$

By $P_v^{(\alpha,\beta)}(x)$ $(v = 0,1,\ldots)$ we denote the Jacobi's polynomials, i.e. algebraic polynomials of degree v orthogonal with the weight function $(1-x)^\alpha(1+x)^\beta$ on the segment $[-1,1]$ and normed by the condition $P_v^{(\alpha,\beta)}(1) = 1$ $(v, 0, 1, \ldots)$.

Denote by $a_n(f)$ the Fourier-Jacobi coefficients of a function f, integrable with the weight funkction $(1-x)^2$ on the segment $[-1,1]$, with respect to the system of Jacobi polynomials $\left\{P_n^{(2,2)}(x)\right\}_{n=0}^{\infty}$, i.e.

$$a_n(f)=\int_{-1}^{1} f(x)P_n^{(2,2)}(x)(1-x^2)^2dx \quad (n=0,1,\dots).$$

The following properties of the operator $\hat{T}(f,x)$ are proved in [1].

Theorem A. Operator T_y has the following properties

1) The operator $T_y(f,x)$ is linear on f,

2) $T_1(f,x)=f(x)$;

3) $$T_y(P_n^{(2,2)},x)=P_n^{(2,2)}(x)\,P_n^{(2,2)}(y)\,T_y(P_n^{(2,2)},x)= \\ =P_n^{(2,2)}(x)\,R_n(y) \quad (n=0,1,\dots)$$

where $R_n(y)=P_{n+2}^{(0,0)}(y)+\frac{3}{2}(1-y^2)P_n^{(2,2)}(y)$;

4) $T_y(1,x)=1$;

5) $a_k\big(T_y(f,x)\big)=R_k(y)a_k(f) \quad (k=0,1,\dots)$.

The following theorem is known as the theorem of coincidence.

Theorem B. Let given numbers p, α and λ be such that $1\le p\le\infty$, $0<\lambda<2$;

$$\begin{aligned} &\frac{1}{2}<\alpha\le 1 && \text{for } p=1 \\ 1-&\frac{1}{2p}<\alpha<\frac{3}{2}-\frac{1}{2p} && \text{for } 1<p<\infty \\ &1\le\alpha<\frac{3}{2} && \text{for } p=\infty. \end{aligned}$$

Let $f\in L_{p\alpha}$. Then

$$E_n(f)_{p\alpha}\le\frac{C_1}{n^{\lambda}}$$

if and only if

$$\hat{\omega}(f,\delta)_{p,\alpha}\le C_2\delta^{\lambda},$$

where the constants C_1 and C_2 do not depend on f, n and δ.

Theorem is proved in [[2]].

2. The inverse theorem.

New we formulate our result

Theorem 2.1. Let given numbers p, α and λ be such that $1 \le p \le \infty$, $0 < \lambda < 2$;

$$\begin{array}{ll} \frac{1}{2} < \alpha \le 1 & \text{for } p = 1 \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \text{for } 1 < p < \infty \\ 1 \le \alpha < \frac{3}{2} & \text{for } p = \infty . \end{array}$$

If $f \in L_{p,\alpha}$, then the following inequality holds

$$\hat{\omega}\left(f, \frac{1}{n}\right)_{p,\alpha} \le C \frac{1}{n^2} \sum_{v=1}^{n} v E_v(f)_{p,\alpha} ,$$

where the constant C does not depend on f and n.

Proof. Let $P_n(x)$ be the polynomial of degree not greater than $n-1$ such that

$$\|f - P_n\|_{p,\alpha,\beta} = E(f)_{p,\alpha,\beta} \quad (n = 1, 2, \ldots) ,$$

and

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \ldots),$$

$Q_0(x) = P_1(x)$.

For given n we chose the natural number N such that

$$\frac{n}{2} < 2^N \le n + 1 .$$

By the proof of Theorem B given in [[1]] it follows that

$$\hat{\omega}\left(f, \frac{1}{n}\right)_{p,\alpha} \le C_1 \left(E_{2^N} + \frac{1}{n^2} \sum_{\mu=0}^{N} 2^{2\mu} \|Q_k\|_{p\alpha} \right) \le$$

$$\le 2C_1 \left(E_{2^N} + \frac{1}{n^2} \sum_{\mu=1}^{N} 2^{2\mu} \big(E_{2^N}(f)_{p,\alpha} + E_{2^{\mu-1}}(f)_{p,\alpha} \big) \right) \le$$

$$\leq 4C_1\left(E_{2^N}+\frac{1}{n^2}\sum_{\mu=0}^{N-1}2^{2(\mu+1)}E_{2^\mu}(f)_{p,\alpha}\right)\leq$$

$$\leq C_2\,\frac{1}{n^2}\sum_{\mu=0}^{N}2^{2(\mu+1)}E_{2^\mu}(f)_{p,\alpha}\,.$$

Considering that for $\mu\geq 1$ we have

$$\sum_{v=2^{n-1}}^{2^\mu-1}vE_v(f)_{p,\alpha}\geq E_{2^\mu}(f)_{p,\alpha}2^{2(\mu-1)}\,,$$

it follows that

$$\hat{\omega}\left(f,\frac{1}{n}\right)_{p,\alpha}\leq C_3\frac{1}{n^2}\left(2^2E_1(f)_{p,\alpha}+\sum_{\mu=1}^{N}\sum_{v=2^{\mu-1}}^{2^\mu-1}vE_v(f)_{p,\alpha}\right)\leq$$

$$\leq C_4\frac{1}{n^2}\sum_{v=1}^{n}vE_v(f)_{p,\alpha}\,.$$

Theorem is proved.

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ЕДНА ЗАБЕЛЕШКА ЗА ИНВЕРЗНАТА ТЕОРЕМА ЗА ОБОПШТЕНИОТ МОДУЛ НА ГЛАТКОСТ

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Р е з и м е

Во трудот се разгледува теорема инверзна на теоремата на Џексон за модулот на глаткост од прв ред генерализирана со средина од асиметричен оператор на обопштени транслации.

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