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FREE $(\mathsf{V};n,m)$-GROUPOIDS

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Abstract. The class of $(\mathsf{V};n,m)$-groupoids, where V is a nontrivial variety of semigroups, and n,m are positive integers, is defined in [5]. Here we give a convenient description of free $(\mathsf{V};n,m)$-groupoids. Almost all usual properties of "freeness" hold, but some exceptions also arise. For example, if $m \geq 2$ and if V is the variety of commutative semigroups, then the set of automorphisms of a free $(\mathsf{V};n,m)$-groupoid with basis B, which induce the identity transformation on B, is infinite.

1. Preliminaries. We give first a list of definitions and results needded in what follows.

1.1. A class V of semigroups is called a variety of semigroups if it admits a system of axioms, every one of which is a semigroup identity, i.e. formula of the form

$$x_{i_1}x_{i_2}\ldots x_{i_p} = x_{j_1}x_{j_2}\ldots x_{j_q} \tag{1.1}$$

where $x_1,x_2,\ldots$ is an infinite set of variables, and i_ν, j_λ, p, q are positive integers. We say that a formula of the form (1.1) holds in V iff it is a consequence of the axiom system of V or, equivalently, iff (1.1) is an identity in every semigroup of V. A variety V is said to be nontrivial if the identity x=y (i.e. $x_1=x_2$) does not hold in V, i.e. if V contains semigroups with more than one element.

Further on in this paper we assume that any considered variety is nontrivial.

1.2. Denote by $\mathbb{N}=\{1,2,..\}$ the set of positive integers, and by $\mathbb{N}_t$ the set $\{1,2,\ldots,t\}$, where $t\in\mathbb{N}$. We consider identities of V of the form (1.1), assuming that $p = q = t$.

The content of a sequence $\underline{i}=(i_1,i_2,\ldots,i_t)$ of positive integers is said to be the set

$$c(\underline{i}) = \{i_1, i_2, \ldots, i_t\} = \{s \mid s = i_\nu \text{ for some } \nu \in N_t\}.$$

Denote by $(\underline{i};V)$ the collection of all $\underline{j}=(j_1,\ldots,j_t)$ such that (1.1) holds in V, and by $\{\underline{i};V\}$ the following collection of sets:

$$\{c(\underline{j}) \mid \underline{j} \in (\underline{i};V)\}.$$

Then one and only one of the following two statements hold (see [5]):

a) $\{k\}$ is an element of $\{\underline{i};V\}$, for every $k \in N$.

b) There exists the least member of $\{\underline{i};V\}$.

This result suggests the following definition of the essential content of $\underline{i}$ in V, which we denote by $\underline{ec}(\underline{i})$. Namely, $\underline{ec}(\underline{i})=\emptyset$ iff a) holds, and in the case b) $\underline{ec}(\underline{i})$ denote the least member of the set $\{\underline{i};V\}$. Note that $\underline{j} \in (\underline{i};V)$ implies $\underline{ec}(\underline{i})=\underline{ec}(\underline{j})$.

Assume now that $i_\lambda=\lambda$, i.e. $\underline{i}=(1,2,\ldots,t)$. Then we say that V is t-regular iff $|\{\underline{i};V\}|=1$, i.e. iff $\underline{ec}(\underline{i})=N_t$, and we say that V is t-nilpotent iff $\underline{ec}(\underline{i})=\emptyset$. Note that a variety V is t-regular iff for each identity of V of the form $x_1\ldots x_t = x_{j_1}\ldots x_{j_t}$ we have that $j: \lambda \to j_\lambda$ is a permutation of N_t. Also, a variety V is t-nilpotent iff $x_1x_2\ldots x_t = x_{t+1}\ldots x_{2t}$ (or, equivalently, $x_1 \cdot x_2 \cdot \ldots \cdot x_t = y^t$) is an identity of V.

A variety V is said to be t-nil iff $x^t=y^t$ is an identity of V. For every t-nil variety V there exists the largest number $\ell \in N_t$ such that if $\underline{i}=(i_1,\ldots,i_t)$ is such that $i_\lambda \in N_t$, $|\{i_1,\ldots,i_t\}|=\ell$, then $\underline{ec}(\underline{i})=\emptyset$. Then we also say that V is (ℓ,t)-nilpotent. Note that if $\ell' \leq \ell$ and V is (ℓ,t)-nilpotent, then V is (ℓ',t)-nilpotent, as well.

1.3. If Q is a nonempty set then we denote by $V(Q)$ a free semigroup in V with a basis Q, and we use the same notation for the semigroup and its carrier. Thus, Q is a generating subset of $V(Q)$ and if $a_\lambda \in Q$ are such that $a_\lambda = a_\nu \Rightarrow \lambda=\nu$, then the equality

$$a_{i_1}a_{i_2}\ldots a_{i_p} = a_{j_1}a_{j_2}\ldots a_{j_q}$$

holds in $V(Q)$ iff (1.1) is an identity of V. For every positive integer t a subset $V_t(Q)$ of $V(Q)$ is defined by

$$V_t(Q) = \{a_1 a_2 \ldots a_t \mid a_i \in Q\}.$$

The nontriviality of V implies $Q = V_1(Q)$, and, moreover,

$$V(Q) = \bigcup \{V_t(Q) \mid t \geq 1\}.$$

For every mapping $\xi: Q \to Q'$ there exists a unique homomorphism $V(\xi): V(Q) \to V(Q')$ which is an extension of ξ, and if $t \geq 1$, then $V(\xi)$ induces a mapping $V_t(\xi): V_t(Q) \to V_t(Q')$.

The notion of the essential content of a sequence $\underline{i} = (i_1, \ldots, i_t)$ of positive integers suggests a corresponding definition of the essential content of an element $u \in V_t(Q)$, which is a subset of Q denoted by $\underline{ec}(u)$. Namely, if $u \in V_t(Q)$ then we put $\underline{ec}(u) = \emptyset$ iff $|Q| \geq 2$ and $u = b^t$ for every $b \in Q$, and in the opposite case $\underline{ec}(u) = \{a_1, \ldots, a_t\}$ iff $u = a_1 \ldots a_t$ in $V_t(Q)$ and for every $b_\lambda \in Q$ such that $u = b_1 \ldots b_t$ it follows $\{a_1, \ldots, a_t\} \subseteq \{b_1, \ldots, b_t\}$.

1.4. Let n and m be two positive integers. If Q is a nonempty set and f a mapping from $V_n(Q)$ into $V_m(Q)$, then we say that $\underline{Q} = (Q; f)$ is a $(V; n, m)$-*groupoid*, Q is the *carrier* and f the *operation* of $\underline{Q}$.

If $\underline{Q} = (Q; f)$ and $\underline{Q}' = (Q'; f')$ are $(V; n, m)$-groupoids, then a mapping $\xi: Q \to Q'$ is said to be a homomorphism from $\underline{Q}$ into $\underline{Q}'$ iff

$$V_m(\xi) f = f' V_n(\xi).$$

If $\xi: Q \to Q'$ is a bijective homomorphism, then $\xi^{-1}: Q' \to Q$ is a homomorphism too (see [5]) and then we say that ξ is an *isomorphism*. The notions "endomorphism", "automorphism" have the usual meanings.

Let $\underline{Q} = (Q; f)$ be a $(V; n, m)$-groupoid and $B \subseteq Q$. The subset B is said to be *free* in $\underline{Q}$ iff for every $(V; n, m)$-groupoid $\underline{Q}' = (Q'; f')$ and every mapping $\xi: B \to Q'$ there is a homomorphism $\psi: \underline{Q} \to \underline{Q}'$ which is an extension of ξ.

If $\underline{Q} = (Q; f)$ is a $(V; n, m)$-groupoid and P is a nonempty subset of $\underline{Q}$, then P is said to be a *subgroupoid* of $\underline{Q}$ iff $f(V_n(P)) \subseteq V_m(P)$. Equivalently, P is a subgroupoid of $\underline{Q}$ iff $\underline{P} = (P; f_P)$ is a $(V; n, m)$-groupoid, where f_P is the restriction of f on $V_n(P)$; then the embedding of P into Q is a homomorphism.

A nonempty intersection of subgroupoids of a $(V;n,m)$-groupoid $\underline{Q}=(Q;f)$ is a subgroupoid as well (see [5]), and thus every nonempty subset B of Q generates a unique subgroupoid $\langle B\rangle$ of $\underline{Q}$; if $\langle B\rangle=\underline{Q}$ then B is called a *generating subset* of $\underline{Q}$.

A $(V;n,m)$-groupoid $\underline{Q}=(Q;f)$ is said to be *free* iff there is a generating subset B of Q which is free in $\underline{Q}$. Then B is called a *basis* of $\underline{Q}$.

2. *$(V;n,m)$-groupoids when V is t-nilpotent*. Here we assume that V is a nontrivial t-nilpotent variety of semigroups, where t=m or t=n, and n,m are positive integers. First we consider the case t=m, i.e. V is m-nilpotent; then $m \geq 2$, $x_1\dots x_m=x_{m+1}\dots x_{2m}$ is an identity of V, and $|V_m(Q)|=1$ for every nonempty set Q.

Proposition 2.1. If Q is a nonempty set, then there is a unique $(V;n,m)$-groupoid $\underline{Q}=(Q;f)$.

Proof. If $u,v\in V_n(Q)$, then $f(u)=b_1\dots b_m=f(v)$, for every sequence $(b_1,\dots,b_m)$ of elements of Q. ∎

The proofs of the following statements are also clear.

Proposition 2.2. Every nonempty subset P of a $(V;n,m)$-groupoid $\underline{Q}=(Q;f)$ is a subgroupoid of $\underline{Q}$. ∎

Proposition 2.3. If $\underline{Q}$ and $\underline{Q}'$ are $(V;n,m)$-groupoids then every mapping $\xi: Q \to Q'$ is a homomoprhism from $\underline{Q}$ into $\underline{Q}'$. ∎

Proposition 2.4. Every $(V;n,m)$-groupoid $\underline{Q}$ is free, and Q is the unique basis of $\underline{Q}$. ∎

Now we consider n-nilpotent varieties which are not m-nilpotent. If $n \leq m$, every n-nilpotent variety is m-nilpotent as well, and so we suppose $n > m \geq 1$.

Note that in this case $|V_n(Q)|=1$ for any nonempty set Q; put $V_n(Q)=\{0\}$. Thus, if $\underline{Q}=(Q;f)$ is a $(V;n,m)$-groupoid, then f is uniquely determined by an element $u=f(0)\in V_m(Q)$. Therefore, we can give the following interpretation of $(V;n,m)$-groupoids, homomorphisms and subgroupoids.

Proposition 2.5. There is a bijection between the class of all $(V;n,m)$-groupoids $(Q;f)$ and the class of all ordered pairs $(Q;u)$, where $u \in V_m(Q)$. ⌑

Thus, we can say that $(Q;u)$ is a $(V;n,m)$-groupoid.

Proposition 2.6. Let $\underline{Q}=(Q;u)$ and $\underline{Q}'=(Q';u')$ be $(V;n,m)$-groupoids and $\xi\colon Q \to Q'$ a mapping. Then ξ is a homomorphism from $\underline{Q}$ into $\underline{Q}'$ iff $V(\xi)(u)=u'$. ⌑

Proposition 2.7. Let $\underline{Q}=(Q;u)$ be a $(V;n,m)$-groupoid and let $U=\underline{ec}(u)$ be the essential content of u. Then a nonempty subset P of Q is a subgroupoid of $\underline{Q}$ iff $U \subseteq P$. Thus $\langle B\rangle = B \cup U$ for every nonempty subset B of Q. ⌑

Proposition 2.8. Let $(Q;u)$ and $(Q';u')$ be $(V;n,m)$-groupoids such that $U=\underline{ec}(u)$, $U'=\underline{ec}(u')$. Then there exists a homomorphism from $(Q;u)$ into $(Q';u')$ iff $|U'| \le |U|$.

Proof. Let $U=\{a_{i_1},\dots,a_{i_m}\}$, $U'=\{b_{j_1},\dots,b_{j_m}\}$, $u=a_{i_1}\dots a_{i_m}$, $u'=b_{j_1}\dots b_{j_m}$. Then $\xi\colon Q \to Q'$ is a homomorphism iff $\xi(a_{i_1})\dots\xi(a_{i_m})=$ $=b_{j_1}\dots b_{j_m}$ holds in $V_m(Q)$, which implies $|U'| \le |\{\xi(a_{i_1}),\dots$ $\dots,\xi(a_{i_m})\}| \le |U|$. ⌑

Now we can give a complete description of free $(V;n,m)$-groupoids, when V is n-nilpotent.

Proposition 2.9. Let $\underline{Q}=(Q;u)$ be a $(V;n,m)$-groupoid, such that $U=\underline{ec}(u)$. $\underline{Q}$ is free iff $|U|=|\underline{ec}(1,2,\dots,m)|$ and then $B=Q\backslash U$ is the unique basis of $\underline{Q}$.

(We say that $|U| = |\underline{ec}(1,2,\dots,m)|$ is the rank of the free $(V;n,m)$-groupoid $\underline{Q}$.)

Proof. It is clear that $|U| \le |\underline{ec}(1,2,\dots,m)|$, and there is $u' \in V_m(Q)$ such that $|\underline{ec}(u')|=|\underline{ec}(1,2,\dots,m)|$. Now, the result follows by Pr. 2.7 and Pr. 2.8. ⌑

Proposition 2.10. Two free $(V;n,m)$-groupoids are isomorphic iff they have equivalent carriers and a same rank. ⌑

Proposition 2.11. Every subgroupoid $\underline{P}$ of a free $(\mathcal{V};n,m)$-groupoid $\underline{Q}=(Q;u)$ is free, and $\underline{P}$ and $\underline{Q}$ have a same rank. ¤

R e m a r k. Similar results can be obtained when (ℓ,n)-nilpotent varieties $\mathcal{V}$ are considered, in the case when Q is a nonempty set such that $|Q| \le \ell$. Namely, in this case each $(\mathcal{V};n,m)$-groupoid $\underline{Q}=(Q;f)$ can be interpreted as an ordered pair $(Q;u)$, where $u\in\mathcal{V}_m(Q)$. Then statements, corresponding to Pr. 2.9, Pr. 2.10 and Pr. 2.11, are also true.

3. $(\mathcal{V};n,m)$-groupoids with free subsets. In what follows we denote by $\mathcal{V}$ a nontrivial variety, by B a nonempty set and by n,m positive integers.

Define a sequence of sets $(B_\alpha \mid \alpha \ge 0)$ as follows:

$$B_0=B,\ B_{\alpha+1}=B_\alpha \cup \mathbb{N}_m \times \mathcal{V}_n(B_\alpha) \tag{3.1}$$

and denote by $[B]$ $(=[B;\mathcal{V};n,m])$ the union of $(B_\alpha \mid \alpha \ge 0)$, i.e.

$$[B] = \bigcup (B_\alpha \mid \alpha \ge 0) \tag{3.2}$$

Define a $(\mathcal{V};n,m)$-groupoid $\underline{[B]}=([B];f)$ by

$$f(x) = (1,x)\dots(m,x) \tag{3.3}$$

for every $x\in\mathcal{V}_n([B])$.

A mapping $\chi\colon \mathcal{V}([B]) \to \mathbb{N}$ is defined as follows:

$$u\in\mathcal{V}(B) \iff \chi(u) = 0,$$
$$u\in\mathcal{V}(B_{\alpha+1})\setminus\mathcal{V}(B_\alpha) \iff \chi(u) = \alpha+1,$$

and we refer to χ as a hierarchy. As a consequence of the definition of χ we have

$$b\in B \Rightarrow \chi(b) = 0,$$
$$u\in\mathcal{V}_n([B]) \Rightarrow \chi((i,u)) = 1 + \chi(u).$$

(Note that if $\mathcal{V}$ is n-nilpotent, then $B_\alpha=B_1$ for every $\alpha \ge 1$, i.e. $[B]=B_1=B\cup\{(1,0),(2,0),\dots,(m,0)\}$, where $\{0\}=\mathcal{V}_n([B])$, and in this case

$$\chi(u) = 0 \iff u\in\mathcal{V}(B),\ \chi(u)=1 \iff u\in\mathcal{V}([B])\setminus\mathcal{V}(B)).$$

Proposition 3.1. B is a free subset of $[B]$.

Proof. Let $\underline{Q}'=(Q';f')$ be a $(V;n,m)$-groupoid, and consider a mapping $\xi: B \to Q'$. We should show that there is a homomorphism $\psi: \underline{[B]} \to \underline{Q}'$ which is an extension of ξ. The construction of ψ will be done by induction on hierarchy.

We put $\xi_o=\xi$ and define ξ_1 as an extension of ξ_o as follows. If $(i,x)\in B_1\setminus B_o$, then

$$f(x) = (1,x)\ldots(m,x),\ f'(V(\xi_o)(x)) = a'_1\ldots a'_m,$$

for some $a'_\lambda \in Q'$. We fix such a sequence $(a'_1,\ldots,a'_m)$ of elements of Q' and define $\xi_1((j,x))$ by $\xi_1((j,x))=a'_j$, for every $j\in\mathbb{N}_m$. Then we have

$$V(\xi_1)(f(x)) = f'(V(\xi_o)(x))$$

for every $x\in V_n(B_o)$.

Suppose that a sequence $\xi_o,\xi_1,\ldots,\xi_\gamma$ of mappings $\xi_\alpha: B_\alpha \to Q'$ is defined in such a way that:

a) $\xi_{\alpha+1}$ is an extension of ξ_α,

b) $V(\xi_{\alpha+1})f = f'V(\xi_\alpha)$

for each $\alpha < \gamma$.

Now, an extension $\xi_{\gamma+1}$ of ξ_γ can be defined such that a) and b) hold for each $\alpha \le \gamma$. Namely, if $u\in B_{\gamma+1}\setminus B_\gamma$, then $u=(i,x)$ for some uniquely determined $i\in\mathbb{N}_m$ and $x\in V_n(B_\gamma)$, where $\chi(x)=\gamma$. (Note that then $(j,x)\in B_{\gamma+1}\setminus B_\gamma$, for every $j\in\mathbb{N}_m$.) We have $f'(V(\xi_\gamma)(x))=c'_1\ldots c'_m\in V_m(Q')$, and we fix such a sequence $(c'_1,\ldots,c'_m)$ of elements of Q. Now, for every $j\in\mathbb{N}_m$, we put $\xi_{\gamma+1}((j,x))=c'_j$, i.e. $\xi_{\gamma+1}(u)=c'_i$. In such a way a mapping $\xi_{\gamma+1}: B_{\gamma+1} \to Q'$ is defined and a) and b) hold for each $\alpha \le \gamma$. Therefore, a chain of mappings $(\xi_\alpha \mid \alpha \ge 0)$ satisfying a) and b) for each $\alpha \ge 0$ is obtained. Its union $\psi=\bigcup(\xi_\alpha \mid \alpha\ge 0)$ is the desired homomorphism from $\underline{[B]}$ into $\underline{Q}'$, which is an extension of ξ. ¤

Let $[[B]]$ be the subgroupoid of $\underline{[B]}$ generated by B, and denote by g the restriction of f on $[[B]]$. As a corollary of *Pr. 3.1* we have:

Proposition 3.2. $\underline{[[B]]}=([[B]];g)$ is a free $(V;n,m)$-groupoid with a basis B. ⌑

Below we give an explicit description of the free $(V;n,m)$-groupoid $([[B]];g)$, for different kinds of varieties V.

If V is m-nilpotent, then $[[B]]=B$, and $(B;g)$ is the unique $(V;n,m)$-groupoid on B.

Assume now that V is not m-nilpotent, i.e. the set $I=\underline{ec}(1,2,\ldots,m)$ is nonempty. This implies that there exists an $\underline{i}=(i_1,i_2,\ldots,i_m)$ such that $I=\{i_1,\ldots,i_m\}\subseteq N_m$ and

a) $x_{i_1}\ldots x_{i_m} = x_1\ldots x_m$ is an identity of V,

b) if $x_{j_1}\ldots x_{j_m} = x_1\ldots x_m$ is an identity of V and $\{j_1,\ldots,j_m\}=J\subseteq N_m$, then $I\subseteq J$.

The following property is true:

Proposition 3.3. Let $(C_\alpha \mid \alpha \geq 0)$ be a collection of sets defined by

$$C_0 = B,\ C_{\alpha+1} = C_\alpha \cup I\times V_n(C_\alpha)$$

Then:

$$[[B]] = \bigcup (C_\alpha \mid \alpha \geq 0),\ \text{and}$$

$$g(x) = (i_1,x)\ldots(i_m,x)$$

for any $x\in V_n([[B]])$. ⌑

Now, if V is n-nilpotent and $n > m$, then $\underline{[[B]]}=(B\cup U;u)$, where $V_n([[B]])=\{0\}$, $u=(i_1,0)\ldots(i_m,0)$, $U=\{(i_1,0),\ldots,(i_m,0)\}$. The same result is true in the case when V is (ℓ,n)-nilpotent and $|B|+|I| \leq \ell$.

Clearly, if V is m-regular, then $[[B]]=[B]$, and so $\underline{[[B]]}=[\underline{B}]$.

4. Some properties of $\underline{[[B]]}$. Using induction on hierarchy, we will show the following statement.

Proposition 4.1. If ξ is an endomorphism of $\underline{[[B]]}$ such that $\xi(b)=b$ for every $b\in B$, then ξ is an automorphism of $\underline{[[B]]}$.

Proof. If V is m-nilpotent, then $[[B]]=B$. So, let $I=\{i_1,\ldots,i_m\}=\underline{ec}(1,2,\ldots,m)\neq\emptyset$. Then, for any $b_\lambda\in B$ we have $\xi(b_\lambda)=b_\lambda$, and since ξ is an endomorphism and $f(b_1\ldots b_n)= =(i_1,\underline{b})\ldots(i_m,\underline{b})$, where $\underline{b}=b_1\ldots b_n$, we have

$$\xi(i_1,\underline{b})\ldots\xi(i_m,\underline{b}) = (i_1,\underline{b})\ldots(i_m,\underline{b}).$$

The last equality implies

$$\underline{ec}(\xi(i_1,\underline{b})\ldots\xi(i_m,\underline{b})) = \underline{ec}(i_1,\underline{b})\ldots(i_m,\underline{b})) = \{(i_1,\underline{b}),\ldots,(i_m,\underline{b})\},\text{ i.e.}$$

$$\{(i_1,\underline{b}),\ldots,(i_m,\underline{b})\}\subseteq\{\xi(i_1,\underline{b}),\ldots,\xi(i_m,\underline{b})\}.$$

This means that the restriction of ξ on $C_1=B\cup I\times V_n(B)$ is a permutation of C_1. We now proceed by induction on hierarchy, using the same arguments as above. Π

Proposition 4.2. Let V be not an m-nilpotent variety and let $x_{i_1}\ldots x_{i_m}=x_{j_1}\ldots x_{j_m}$ be an identity of V, such that $I=\{i_1,\ldots,i_m\}=\{j_1,\ldots,j_m\}=\underline{ec}(1,2,\ldots,m)$ and $\pi: i_\nu \to j_\nu$ is a permutation of I, $\pi\neq 1_I$. Then there exists an automorphism ξ of $[[B]]$ such that $\xi(b)=b$ for every $b\in B$ and ξ is not the identity automorphism of $[[B]]$.

Proof. Denote by ξ_o the identity permutation of $C_o=B$, and let $H_{\alpha+1}=C_{\alpha+1}\setminus C_\alpha$, for every $\alpha\geq 0$. Define a mapping $\xi_1: C_1\to C_1$ by: $b\in B \Rightarrow \xi_1(b)=\xi_o(b)$, $x\in V_n(C_o) \Rightarrow \xi_1((i_\lambda,x))=(j_\lambda,x)$, for every $i_\lambda\in I$. Then ξ_1 is a nontrivial permutation of C_1 which extends ξ_o, such that ξ_1 is a permutation of H_1.

Suppose that $\xi_o,\xi_1,\ldots,\xi_\alpha$ are defined permutations of $C_o,C_1,\ldots,C_\alpha$ such that $\xi_{\beta+1}$ is an extension of ξ_β and $\xi_{\beta+1}$ is a permutation of $H_{\beta+1}$. Define a mapping $\xi_{\alpha+1}: C_{\alpha+1}\to C_{\alpha+1}$ by:

$$u\in C_\alpha \Rightarrow \xi_{\alpha+1}(u) = \xi_\alpha(u).$$

If $u\in H_{\alpha+1}$, then $u=(i_\lambda,x)$ for some $i_\lambda\in I$, $x\in V_n(C_\alpha)$ with $\chi(x)=\alpha$, and then we put

$$\xi_{\alpha+1}(u) = \xi_{\alpha+1}((i_\lambda,x)) = (j_\lambda,V_n(\xi_\alpha)(x)).$$

Note that $\xi_{\alpha+1}(u)\in H_{\alpha+1}$, i.e. $\xi_{\alpha+1}$ is a permutation of $H_{\alpha+1}$. Now, since ξ_α is a permutation of C_α, we have that $V_n(\xi_\alpha)$ is a permutation of $V_n(C_\alpha)$, which implies that $\xi_{\alpha+1}$ is a (nontrivial) permutation of $C_{\alpha+1}$.

In such a way a chain of permutations $(\xi_\alpha \mid \alpha \geq 0)$ is obtained, and for every $\alpha \geq 0$ and each $x\in V_n(C_\alpha)$ the following property is satisfied:

$$\begin{aligned} V_m(\xi_{\alpha+1})(g(x)) &= V_m(\xi_{\alpha+1})((i_1,x)\dots(i_m,x)) = \\ &= (j_1,V_n(\xi_\alpha)(x))\dots(j_m,V_n(\xi_\alpha)(x)) = \\ &= (i_1,V_n(\xi_\alpha)x))\dots(i_m,V_n(\xi_\alpha)(x)) = \\ &= g(V_n(\xi_\alpha)(x)). \end{aligned}$$

If we put $\xi=\bigcup(\xi_\alpha \mid \alpha \geq 0)$, then ξ is a nontrivial permutation of $[[B]]$ such that

$$V_m(\xi)g = gV_n(\xi),$$

i.e. ξ is an automorphism of $\underline{[[B]]}$, different to identity one. ⌑

Proposition 4.2. implies that it is not necessarily true the well known property that "the identity automorphism of a universal algebra is the unique endomorphism which is the extension of the embedding of a generating subset into the algebra". Note, for example, that the conditions of Pr. 4.2 are satisfied for every subvariety V of the variety of commutative semigroups.

The next statement is one of the main results of this paper.

Proposition 4.3. If $\underline{Q}=(Q;f)$ is a free $(V;n,m)$-groupoid with a nonempty basis B, then $(Q;f)$ is isomorphic with $\underline{[[B]]}$.

Proof. Let $\underline{Q}$ be free. Then there are homomorphisms $\xi\colon \underline{Q} \to \underline{[[B]]}$ and $\psi\colon \underline{[[B]]} \to \underline{Q}$ such that $\xi(b)=\psi(b)=b$ for every $b\in B$. Therefore, $\zeta=\xi\psi$ is an endomorphism of $\underline{[[B]]}$ such that $\zeta(b)=b$ for every $b\in B$, and by Pr. 4.1 this implies that ζ is an automorphism of $\underline{[[B]]}$. Thus, ψ is an injection and $P=\psi([[B]])$ is a subgroupoid of $\underline{Q}$ such that $B\subseteq P$. Now, $P=Q$, since B is a generating subset of Q, i.e. ψ is surjective as well. ⌑

As a corollary of Pr. 4.3 we have:

Proposition 4.4. $\underline{Q}$ is a free (V;n,m)-groupoid with a non-empty basis B iff there is an isomorphism ξ: $\underline{Q} \to \underline{[[B]]}$ such that $\xi(b)=b$ for every $b \in B$. ∎

Proposition 4.5. B is the unique basis of $\underline{[[B]]}$.

Proof. Let $b \in B$. Then $[[B]] \setminus \{b\}$ is a subgroupoid of $\underline{[[B]]}$, which means that B is a subset of every generating subset of $\underline{[[B]]}$.

Assume that D is a basis of $[[B]]$, and hence $B \subseteq D$. If $B \subset D$, take an element $b_o \in B$ and define a mapping ξ: $D \to [[B]]$ by $\xi(b)=b$ for every $b \in B$, $\xi(d)=b_o$ for every $d \in D \setminus B$. Then there exists an endomorphism ψ of $\underline{[[B]]}$ which is an extension of ξ. By Pr. 4.1 we have that ξ is also an automorphism of $\underline{[[B]]}$, but this is a contradiction with the fact: $\psi(d)=b_o=\psi(b_o)$ for some $d \neq b_o$. ∎

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СЛОБОДНИ (V;n,m)-ГРУПОИДИ

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Р е з и м е

Во трудот [5] е дефиниран поимот (V;n,m)-групоид, каде што V е нетривијално многуобразие полугрупи. Овде се опишуваат слободните (V;n,m)-групоиди и се покажува точноста на повеќе стандардни "слободни" својства, но и на некои нестандардни. На пример, ако m ≥ 2 и V е многукратноста од комутативни полугрупи, тогаш множеството автоморфизми на еден слободен (V;n,m)-групоид, коишто го индуцираат идентичното пресликување на базисот, е бесконечно.

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