II.3. Projective com(m+k,m)-semigroups and groups

The complex com(m+k,m)-groupoids $(\mathbb{C},f)$ can be examined via polynomial mappings

$$\phi : P_{m+k} \to P_m$$

such that

$$f(z_1^{m+k})=w_1^m \Leftrightarrow \phi((t-z_1)\dots(t-z_{m+k})) = (t-w_1)\dots(t-w_m).$$

If the coefficients of the polynomial $\phi(\rho)$ (except the coefficient 1 in front of t^m) are linear functions of the coefficients of the polynomial ρ with degree m+k, then the complex com(m+k,m)-groupoid determined in this way was called affine in II.2. The affine com(n,m)-group structures can be generalized in the following way. We can consider the polynomial mappings

$$\phi^* : P^*_{m+k} \to P^*_m$$

instead of the polynomial mappings $\phi:P_{m+k}\to P_m$ where for a positive integer n,

$$P^*_n = \{a_0t^n - a_1t^{n-1} + \dots + (-1)^n a_n \mid (a_0,a_1,\dots,a_n)\in\mathbb{C}P^n\}$$

where $\mathbb{C}P^n$ is the n-dimensional complex projective space.

Further, if $\rho\in P^*_n$ and $\deg(\rho)=r<n$, then we say that ∞ is a root of the polynomial ρ with multiplicity n-r. The converse also holds. Indeed, if the numbers $z_1,\dots,z_n\in\mathbb{C}^* = \mathbb{C}\cup\{\infty\}$ which may not be different, be given, then there exists a unique polynomial $\rho\in P^*_n$ such that $z_1,\dots,z_n$ are roots of ρ. Thus a polynomial mapping $\phi^*:P^*_{m+k}\to P^*_m$ corresponds to each com(m+k,m)-groupoid on $\mathbb{C}^*$, and conversely, a com(m+k,m)- groupoid corresonds to each such mapping.

In general we can write

$$\begin{aligned}&\phi^*(a_0t^{m+k} - a_1t^{m+k-1} + \dots +(-1)^{m+k}a_{m+k})\\ &= F_0t^m - F_1t^{m-1} + F_2t^{m-2} - \dots +(-1)^m F_m \qquad (3.1)\end{aligned}$$

where F_i , $0\le i\le m$, are functions of $a_0,a_1,\dots,a_{m+k}$.

Definition 3.1. If F_i $(i=0,1,\dots,m)$ are linear functions of $a_0,a_1,\dots,a_{m+k}$, i.e.

$$F_i(a_0,a_1,\dots,a_{m+k}) = \sum_{0\le j\le m+k} \alpha_{ij}a_j \qquad (0\le i\le m) \qquad (3.2)$$

then the corresponding com(m+k,m)-groupoid on $\mathbb{C}^*$ which is induced by the polynomial mapping (3.1) is called a **projective** com(m+k,m)-groupoid. If this groupoid is associative, then it is called a **projective** com(m+k,m)-semigroup. A com(m+k,m)-group on a non-empty subset of $\mathbb{C}^*$ which is obtained by removing the singular elements from a projective com(n+k,m)-group is called a **projective** com(m+k,m)-group.

Let a com(m+k,m)-groupoid (semigroup, group) (G,f) be given, and let an arbitrary bijection $\varphi : G \to G$ be given. Then φ induces a com(m+k,m)-groupoid (semigroup, group) (G,f') as follows:

$$f'(z_1^{m+k})=w_1^m \iff$$
$$f(\varphi^{-1}(z_1),\dots,\varphi^{-1}(z_{m+k}))=(\varphi^{-1}(w_1),\dots,\varphi^{-1}(w_m)).$$

These two groupoids (semigroups, groups) (G,f) and (G,f') are isomorphic. If $G=\mathbb{C}^*$ and $(\mathbb{C}^*,f)$ is a projective com(m+k,m)-groupoid, then $(\mathbb{C}^*,f')$ may not be a projective com(m+k,m)-groupoid. In order to study the projective semigroups and groups, the following theorem has an important role.

Theorem 3.1. Let $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary bilinear transformation $\varphi(z)=(az+b)/(cz+d)$ $(ad-bc\ne 0)$, and let $(\mathbb{C}^*,f)$ be a projective com(m+k,m)-groupoid. Then the induced com(m+k,m)-groupoid by φ is projective.

Proof. Suppose that the projective com(m+k,m)-groupoid is given by the linear functions

$$F_i(a_0,a_1,\dots,a_{m+k}) = \sum_{0\le j\le m+k} \alpha_{ij}a_j \qquad (0\le i\le m)$$

and that the induced com(m+k,m)-groupoid is determined by the functions $F_i(a_0,\dots,a_{m+k})$. We will prove that they are linear functions. Since each bilinear transformation can be represented as a composition of the following transformations 1. $\varphi(z)=1/z$, 2. $\varphi(z)=z/\lambda$ $(\lambda\ne 0,\ \lambda\ne\infty)$ and 3. $\varphi(z)=z-\lambda$ $(\lambda\ne\infty)$, it is sufficient to prove the theorem in each of these special cases. Assume that $z_1,\dots,z_{m+k}$, $w_1,\dots,w_m\in\mathbb{C}^*$ are such that $f(z_1^{m+k})=w_1^m$.

1. Let $\varphi(z)=1/z$. We will verify that its inverse transformation $\varphi^{-1}(z) = 1/z$ induces the following mapping:

$$\mathbb{C}P^{m+k} \ni (a_0, a_1, \dots, a_{m+k}) \to (a_{m+k}, a_{m+k-1}, \dots, a_0) \in \mathbb{C}P^{m+k}. \qquad (3.3)$$

First let us suppose that $z_i \neq \infty$ and $z_i \neq 0$ for each $i \in \{0, 1, \dots, m+k\}$. Then $a_0 \neq 0$, $a_{m+k} \neq 0$ and $a_i/a_0 = \wp^i(z_1^{m+k})$, for $(i \in \{1, \dots, m+k-1\})$. Hence we obtain

$$(\wp^i)' = \wp^i(\varphi^{-1}(z_1), \dots, \varphi^{-1}(z_{m+k})) = \wp^i(z_1^{-1}, \dots, z_{m+k}^{-1})$$

$$= \wp^{m+k-i}(z_1^{m+k}) / \wp^{m+k}(z_1^{m+k})$$

$$= (a_{m+k-i}/a_0)/(a_{m+k}/a_0) = (a_{m+k-i}/a_{m+k}),$$

and thus, in $\mathbb{C}P^{m+k}$:

$$(1, \wp^{1'}, \wp^{2'}, \dots, \wp^{m+k'}) = (a_{m+k}, a_{m+k-1}, \dots, a_0).$$

Herewith we have proved that

$$(a_0, a_1, \dots, a_{m+k}) \to (a_{m+k}, a_{m+k-1}, \dots, a_0)$$

if $z_i \neq \infty, 0$ for each $i \in \{0, \dots, m+k\}$. Now let us suppose that among the numbers $z_1, \dots, z_{m+k}$ there exist exactly r numbers which are equal to ∞, and the others are not equal to zero. Without loss of generality we can assume that $z_{m+k} = z_{m+k-1} = \dots = z_{m+k+1-r} = \infty$. Then $a_0 = a_1 = \dots = a_{r-1} = 0$, $a_r \neq 0$ and

$$a_{r+i}/a_r = \wp^i(z_1^{m+k-r}) \qquad (i \in \{1, 2, \dots, m+k-r\}).$$

Hence we obtain

$$\wp^{i'} = \wp^i(z_1^{-1}, \dots, z_{m+k-r}^{-1}, 0, \dots, 0) = \wp^i(z_1^{-1}, \dots, z_{m+k-r}^{-1}) =$$

$$= \wp^{m+k-r-i}(z_1^{m+k-r}) / \wp^{m+k-r}(z_1^{m+k-r}) = (a_{m+k-i}/a_r)/(a_{m+k}/a_r)$$

$$= a_{m+k-i}/a_{m+k} \quad \text{for } i \in \{1, \dots, m+k-r\} \quad \text{and}$$

$$\wp^{i'} = 0 \quad \text{for} \quad i \in \{m+k-r+1, \dots, m+k\}.$$

Thus, in $\mathbb{C}P^{m+k}$

$$(1, \wp^{1'}, \dots, \wp^{m+k'}) = (a_{m+k}, a_{m+k-1}, \dots, a_{m+k-r}, 0, 0, \dots, 0),$$

and in this case we also have

$$(a_0, \dots, a_{m+k}) \to (a_{m+k}, \dots, a_0).$$

Analogously one can verify this statement in the special case when some of the numbers $z_1, \dots, z_{m+k}$ are zeros. For the sake of simplicity we skip those special cases like $z_i=0$, $z_i=\infty$, $w_i=0$ or $w_i=\infty$ for some i.

Further we obtain

$$\frac{F'_i(a_0,\ldots,a_{m+k})}{F'_0(a_0,\ldots,a_{m+k})}=\varphi^i(w_1^m)=\frac{\varphi^{m-i}(\varphi^{-1}(w_1^m))}{\varphi^m(\varphi^{-1}(w_1^m))}=\frac{F_{m-i}(a_{m+k},\ldots,a_0)}{F_m(a_{m+k},\ldots,a_0)}$$

for $i\in\{1,\ldots,m\}$. The equality implies that

$$F'_i(a_0,\ldots,a_{m+k}) = F_{m-i}(a_{m+k},\ldots,a_0) =$$

$$= \sum_{0\le j\le m+k} \alpha_{(m-i)j}\cdot a_{m+k-j} = \sum_{0\le j\le m+k} \alpha_{(m-i)(m+k-j)}\cdot a_j$$

for $i\in\{0,1,\ldots,m\}$. Thus F'_i $(i\in\{0,1,\ldots,m\})$ **are linear** functions of $a_0,a_1,\ldots,a_{m+k}$. Moreover we notice that

$$\alpha'_{ij} = \alpha_{(m-i)(m+k-j)} \tag{3.4}$$

for $i\in\{0,1,\ldots,m\}$ and $j\in\{0,1,\ldots,m+k\}$.

2. Let $\varphi(z)=z/\lambda$ $(\lambda\neq 0,\ \lambda\neq\infty)$. It is easy to verify that in an analogous way as the one in the previous case, the inverse transformation $\varphi^{-1}(z)=\lambda z$ induces the following mapping:

$$\mathbb{C}P^{m+k}\ni(a_0,a_1,\ldots,a_{m+k}) \to (a_0,\lambda a_1,\lambda^2 a_2,\ldots,\lambda^{m+k}a_{m+k})\in\mathbb{C}P^{m+k}.$$

Thus we obtain

$$\frac{F'_i(a_0^{m+k})}{F'_0(a_0^{m+k})} =\varphi^i(w_1^m)=\frac{1}{\lambda^i}\varphi^i(\varphi^{-1}(w_1^m))=\frac{1}{\lambda^i}\frac{F_i(a_0,\lambda a_1,\ldots,\lambda^{m+k}a_{m+k})}{F_0(a_0,\lambda a_1,\ldots,\lambda^{m+k}a_{m+k})}$$

for $i\in\{1,\ldots,m\}$. Hence we obtain

$$F'_i(a_0,a_1,\ldots,a_{m+k})=\lambda^{-i}F_i(a_0,\lambda a_1,\ldots,\lambda^{m+k}a_{m+k})=\lambda^{-i}\sum_{j=0}^{m+k}\alpha_{ij}\lambda^j a_j$$

for $i\in\{0,1,\ldots,m\}$. Thus F'_i are linear functions of $a_0,\ldots,a_{m+k}$ and moreover

$$\alpha'_{ij} = \lambda^{j-i}\alpha_{ij} \tag{3.5}$$

for $i\in\{0,1,\ldots,m\}$ and $j\in\{0,1,\ldots,m+k\}$.

3. Let $\varphi(z)=z-\lambda$ $(\lambda\neq\infty)$. **Analogously to** (3.3), the inverse transformation $\varphi^{-1}(z)=z+\lambda$ **induces the following mapping**:

$$\mathbb{C}P^{m+k}\ni(a_0,a_1,\ldots,a_{m+k}) \to (b_0,b_1,\ldots,b_{m+k})\in\mathbb{C}P^{m+k}$$

where $b_i = \sum_{0\le r\le i}\lambda^r\binom{r+m+k-i}{r}a_{i-r}$ for $i\in\{0,1,\ldots,m+k\}$.

Using this fact, further we obtain

$$\frac{F_i'(a_0,\ldots,a_{m+k})}{F_0'(a_0,\ldots,a_{m+k})}=\varphi^i(w_1^m)=\sum_{0\le r\le i}(-\lambda)^r\binom{r+m-i}{r}\varphi^{i-r}(\varphi^{-1}(w_1^m))=$$

$$\sum_{0\le r\le i}(-\lambda)^r\binom{r+m-i}{r}\frac{F_{i-r}(b_0,b_1,\ldots,b_{m+k})}{F_0(b_0,b_1,\ldots,b_{m+k})}=$$

$$\sum_{0\le r\le i}(-\lambda)^r\binom{r+m-i}{r}\sum_{0\le p\le m+k}\alpha_{(i-r)p}b_p \Big/ \sum_{0\le p\le m+k}\alpha_{0p}b_p$$

for $i\in\{1,\ldots,m\}$. Hence we get

$$F_i'(a_0,\ldots,a_{m+k})=\sum_{r=0}^{i}(-\lambda)^r\binom{r+m-i}{r}\sum_{p=0}^{m+k}\alpha_{(i-r)p}b_p=$$

$$\sum_{r=0}^{i}(-\lambda)^r\binom{r+m-i}{r}\sum_{p=0}^{m+k}\alpha_{(i-r)p}\sum_{s=0}^{p}\lambda^s\binom{s+m+k-p}{s}a_{p-s}=\sum_{j=0}^{m+k}\alpha'_{ij}a_j$$

for $i\in\{0,1,\ldots,m\}$, where

$$\alpha'_{ij}=\sum_{0\le r\le i}(-\lambda)^r\binom{r+m-i}{r}\sum_{j\le p\le m+k}\alpha_{(i-r)p}\lambda^{p-j}\binom{m+k-j}{p-j}.$$

Putting $p=s+j$, we obtain

$$\alpha'_{ij}=\sum_{0\le r\le i}(-\lambda)^r\binom{r+m-i}{r}\sum_{0\le s\le m+k-j}\alpha_{(i-r)(s+j)}\lambda^s\binom{m+k-j}{s}\tag{3.6}$$

where $i\in\{0,1,\ldots,m\}$ and $j\in\{0,1,\ldots,m+k\}$. This finishes the proof of the theorem.■

Now we will introduce a characteristic polynomial for a given projective com(m+k,m)-groupoid on $\mathbb{C}^*$. Let $(\mathbb{C}^*,f)$ be an arbitrary projective com(m+k,m)-groupoid, and let $f(z_1^kx_1^m)=y_1^m$. The axiom for solvability reduces to the following. Suppose that $y_1,\ldots,y_m\in\mathbb{C}^*$ are the roots of the following polynomial

$$F_0t^m-F_1t^{m-1}+F_2t^{m-2}-\ldots+(-1)^mF_m=0$$

where $(F_0,\ldots,F_m)\in\mathbb{C}P^m$, and that $z_1,\ldots,z_k\in\mathbb{C}^*$ are the roots of the following polynomial

$$G_0t^k-G_1t^{k-1}+G_2t^{k-2}-\ldots+(-1)^kG_k=0$$

where $(G_0,\ldots,G_k)\in\mathbb{C}P^k$. The aim is to find numbers $x_1,\ldots,x_m\in\mathbb{C}^*$ as the roots of a polynomial

$$H_0t^m-H_1t^{m-1}+H_2t^{m-2}-\ldots+(-1)^mH_m=0$$

where $(H_0,\ldots,H_m)\in\mathbb{C}P^m$. Then the equation $f(z_1^kx_1^m)=y_1^m$ can be written in the following form:

$$\sum_{0\le t\le n}\ \sum_{0\le r\le \min(t,m)} \alpha_{it}H_rG_{t-r} = F_i \tag{3.7}$$

for each $i\in\{0,1,\dots,m\}$ and where $G_{t-r}=0$ for $t-r>k$. We notice that (3.7) is analogous to (2.26) for the affine com(m+k,m)- groupoids. The equality (3.7) should be conceived as an equality in $\mathbb{C}P^m$ but not in $\mathbb{C}^{m+1}$. Thus the axiom of solvability reduces to the requirement that the system (3.7) has the unique solution $(H_0,H_1,\dots,H_m)\in\mathbb{C}P^m$. The main determinant of the linear system (3.7) is

$$\Delta^*(z_1^k)=\begin{vmatrix} \sum_{i=0}^k \alpha_{0i}G_i & \sum_{i=0}^k \alpha_{0(i+1)}G_i & \dots & \sum_{i=0}^k \alpha_{0(i+m)}G_i \\ \sum_{i=0}^k \alpha_{1i}G_i & \sum_{i=0}^k \alpha_{1(i+1)}G_i & \dots & \sum_{i=0}^k \alpha_{1(i+m)}G_i \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^k \alpha_{mi}G_i & \sum_{i=0}^k \alpha_{m(i+1)}G_i & \dots & \sum_{i=0}^k \alpha_{m(i+m)}G_i \end{vmatrix} \tag{3.8}$$

or shortly,

$$\Delta^*(z_1,\dots,z_k) = \det\Big(\sum_{0\le i\le k}\alpha_{j(i+s)}G_i\Big) \tag{3.8'}$$

where $j,s\in\{0,1,\dots,m\}$. The polynomial (3.8) is called the **characteristic polynomial** for the considered projective com(m+k,m)-groupoid (semigroup, group). Thus the axiom of solvability is satisfied iff the polynomial $\Delta^*(z_1,\dots,z_k)$ never becomes zero. The degree of this polynomial with respect to each argument is $\le m+1$, while the degree of the characteristic polynomial in the affine case was $\le m$. Specially if $k=1$, the characteristic polynomial can be written in the following form:

$$\Delta^*(z)=\begin{vmatrix} (-z)^{m+1} & (-z)^m & (-z)^{m-1} & \dots & 1 \\ \alpha_{00} & \alpha_{01} & \alpha_{02} & & \alpha_{0(m+1)} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(m+1)} \\ - & - & - & - & - \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{m(m+1)} \end{vmatrix}. \tag{3.9}$$

Now we will reveal the connection between the affine and projective group structures. It is given by the following Theorem.

Theorem 3.2. Each affine com(m+k,m)-groupoid can be treated as a com(m+k,m)-subgroupoid of a projective com(m+k,m)-groupoid $(\mathbb{C}^*,f^*)$ where $\infty\in\mathbb{C}^*$ is a singular element. Conversely, let $(\mathbb{C}^*,f^*)$ be an arbitrary projective com(m+k,m)-groupoid such that there exists $\xi\in\mathbb{C}^*$ and $(\mathbb{C}^*\setminus\{\xi\}, f^*)$ is a projective com(m+k,m)-subgroupoid from $(\mathbb{C}^*, f^*)$. Then the subgroupoid $(\mathbb{C}^*\setminus\{\xi\}, f^*)$ is isomorphic to an affine com(m+k,m)-groupoid. Besides that, each projective com(m+k,m)-group is isomorphic to an affine com(m+k,m)-group.

Proof. Let us suppose that the affine com(m+k,m)- groupoid $(\mathbb{C}, f)$ is given by the following functions:

$$F_i(a_1,\dots,a_{m+k}) = \alpha_{i0} + \sum_{1\le r\le m+k} \alpha_{ir} a_r \qquad (i\in\{1,\dots,m\}).$$

If we put $\alpha_{00}=1$ and $\alpha_{0i}=0$ for $i\in\{1,\dots,m+k\}$ we obtain a projective com(m+k,m)-groupoid $(\mathbb{C}^*, f^*)$ which is given by the following functions:

$$F_i^*(a_0,a_1,\dots,a_{m+k}) = \sum_{0\le j\le m+k} \alpha_{ij} a_j \qquad (i\in\{0,\dots,m\}).$$

Now $(\mathbb{C}, f)$ is a com(m+k,m)-subgroupoid of $(\mathbb{C}^*, f^*)$ because $F_i^*(1,a_1,\dots,a_{m+k})=F_i(a_1,\dots,a_{m+k})$ and $F_0^*(1,a_1,\dots,a_{m+k})=1$. Further we shall prove that $\infty\in\mathbb{C}^*$ is a singular element for $(\mathbb{C}^*, f^*)$ i.e. $\varphi_{\infty,\dots,\infty}$ is not a bijection. Let $f^*(\infty,\dots,\infty,z_1^m)=w_1^m$. In that case $a_0=a_1=\dots=a_{k-1}=0$ and so $F_0^*=a_0=0$. Hence at least one of the elements $w_1,\dots,w_m$ is ∞ and $\varphi_{\infty,\dots,\infty}$ is not a bijection. Thus $\infty\in\mathbb{C}^*$ is a singular element for $(\mathbb{C}^*, f^*)$, and the first part of the theorem is proven. Before we prove the remaining part of the theorem, it is convenient to consider the following:

Remark. We notice that according to the definitions of a characteristic polynomial for affine com(m+k,m)-groupoids and projective com(m+k,m)-groupoids, the characteristic polynomials for $(\mathbb{C}^*, f^*)$ and $(\mathbb{C}, f)$ are the same. The characteristic polynomial $\Delta(z_1^k)$ for $(\mathbb{C}, f)$ has degree $\le m$ with respect to each argument, but the characteristic polynomial $\Delta^*(z_1^k)$ for $(\mathbb{C}^*, f^*)$ has degree $\le m+1$ with respect to each argument. The above statement gives us the right to consider $\Delta^*(z_1^k)$ as a polynomial in P_{m+1}^*, for each argument. Indeed, if a polynomial of degree $\le m$ is considered as an element of P_{m+1}^*, then ∞ is a root of it. On the other hand we saw that ∞ is a singular element of $(\mathbb{C}^*, f^*)$.

Now let us return to the proof of the theorem. Let $(\mathbb{C}^*, f^*)$ be a projective com(m+k,m)-groupoid and let $\xi\in\mathbb{C}^*$ be such that $(\mathbb{C}^*\setminus\{\xi\}, f^*)$ is subgroupoid of $(\mathbb{C}^*, f^*)$. We choose an arbitrary bilinear transformation Ψ, such that $\Psi(\xi)=\infty$. According to Theorem 3.1 the com(m+k,m)-groupoid $(\mathbb{C}, f)$ which is induced by the transformation Ψ is projective, and it is isomorphic to $(\mathbb{C}^*\setminus\{\xi\}, f^*)$. We will prove that $(\mathbb{C}, f)$ is an affine com(m+k, m)-groupoid. Indeed, as it is a com(m+k, m)-groupoid on $\mathbb{C}$, we can suppose that $a_0 \equiv 1$ and $F_0 \equiv 1$. Since

$$1 \equiv F_0(1, a_1, \dots, a_{m+k}) = \alpha_{00} + \sum_{1\le i\le m+k} \alpha_{0i} a_i,$$

it follows that $\alpha_{00}=1$ and $\alpha_{0i}=0$ for $i\in\{1,\dots,m+k\}$, which means that $(\mathbb{C}, f)$ is an affine com(m+k, m)-groupoid.

Now let a projective com(m+k, m)-group be given. In fact it is obtained from a non-singular projective com(m+k, m)- semigroup by removing the singular elements. We will prove that the set of singular elements $\mathcal{R}$ is not empty. We know that

$$\mathcal{R} = \{z\in\mathbb{C}^* \mid \Delta^*(z, \dots, z) = 0\}$$

and $\Delta^*(z,\dots,z)$ is a polynomial of degree $k(m+1)$. Thus $\Delta^*(z,\dots,z)=0$ has at least one root. Specially, if $\Delta^*(z,\dots,z)=$const. then $z=\infty$ is the unique root of it. Hence $\mathcal{R}\neq\emptyset$. Let $\theta\in\mathcal{R}$. We choose a bilinear transformation Ψ such that $\Psi(\theta)=\infty$. This bilinear transformation induces a projective com(m+k, m)-group on $\mathbb{C}^*\setminus\Psi(\mathcal{R})\subseteq\mathbb{C}$ which is isomorphic to the given projective com(m+k, m)-group. So for the induced projective com(m+k, m)-group we can put $a_0\equiv 1$ and $F_0\equiv 1$. Hence we obtain that $\alpha_{00}=1$ and $\alpha_{0i}=0$ for $i\in\{1,\dots,m+k\}$, and the induced projective com(m+k, m)-group is affine.∎

The following theorem gives us the necessary and sufficient condition for a projective com(m+k, m)-groupoid on $\mathbb{C}^*$ to be a com(m+k, m)-semigroup.

Theorem 3.3. A projective com(m+k, m)-groupoid on $\mathbb{C}^*$ which is given by the elements α_{ij} $(i\in\{0,1,\dots,m\},\ j\in\{0,1,\dots,m+k\})$ is associative iff

$$\mathcal{A}_s \cdot \mathcal{A}_0 = 0\ , \tag{3.10}$$

for each $s=1,2,\ \dots,\ k$, where

$$\mathcal{A}_s = \begin{bmatrix} \alpha_{0(s-1)} & \alpha_{0s} & \alpha_{0(s+1)} & \cdots & \alpha_{0(s+m)} \\ \alpha_{1(s-1)} & \alpha_{1s} & \alpha_{1(s+1)} & \cdots & \alpha_{1(s+m)} \\ \alpha_{2(s-1)} & \alpha_{2s} & \alpha_{2(s+1)} & \cdots & \alpha_{2(s+m)} \\ \hline \alpha_{m(s-1)} & \alpha_{ms} & \alpha_{m(s+1)} & \cdots & \alpha_{m(s+m)} \end{bmatrix}$$

and

$$\mathcal{A}_0 = \begin{bmatrix} -\alpha_{01} & -\alpha_{02} & \cdots & -\alpha_{0(m+k)} \\ \alpha_{00}-\alpha_{11} & \alpha_{01}-\alpha_{12} & \cdots & \alpha_{0(m+k-1)}-\alpha_{1(m+k)} \\ \alpha_{10}-\alpha_{21} & \alpha_{11}-\alpha_{22} & \cdots & \alpha_{1(m+k-1)}-\alpha_{2(m+k)} \\ \text{-----} & \text{-----} & \text{-----} & \text{-----} \\ \alpha_{m0}-\alpha_{(m+1)1} & \alpha_{m1}-\alpha_{(m+1)2} & \cdots & \alpha_{m(m+k-1)}-\alpha_{(m+1)(m+k)} \end{bmatrix}$$

or shortly,

$$\sum_{-1\le j\le m} \alpha_{i(s+j)}(\alpha_{j(r-1)}-\alpha_{(j+1)r})=0 \qquad (3.10')$$

where $i\in\{0,1,\ldots,m\}$, $r\in\{1,\ldots,m+k\}$, $s\in\{1,\ldots,k\}$ and $\alpha_{(-1)j}=0$ for $j\in\{0,1,\ldots,m+k-1\}$.

Proof. According to Proposition I.3.1 we will prove that the transformations φ_p and φ_q commute for each $p,q\in(\mathbb{C}^*)^{(m)}$ iff (3.10) holds. We know that the transformation φ_p is given by (3.7). Moreover, it is easy to see that it can be written in the following form:

$$A_0G_0 + A_1G_1 + \ \ldots \ + A_kG_k \qquad (3.11)$$

where $(G_0,G_1,\ldots,G_k)\in\mathbb{C}P^k$ is such a k-typle that $p=(p_1,\ldots,p_k)$ and $p_1,\ldots,p_k$ are the roots of the polynomial equation

$$G_0t^k - G_1t^{k-1} + \ \ldots \ +(-1)^kG_k = 0,$$

and

$$A_i = \begin{bmatrix} \alpha_{0i} & \alpha_{0(i+1)} & \cdots & \alpha_{0(i+m)} \\ \alpha_{1i} & \alpha_{1(i+1)} & \cdots & \alpha_{1(i+m)} \\ \text{-----} & \text{-----} & \text{-----} & \text{-----} \\ \alpha_{mi} & \alpha_{m(i+1)} & \cdots & \alpha_{m(i+m)} \end{bmatrix} \qquad (3.12)$$

for $i\in\{0,1,\ldots,k\}$.

Two transformations φ_p and φ_q commute iff the matrices $\bar{\varphi}_p=\sum_{0\le i\le k} A_iG_i$ and $\bar{\varphi}_q=\sum_{0\le i\le k} A_iG_i'$ commute. This is satisfied for arbitrary numbers G_i and G_i' $(i\in\{0,1,\ldots,k\})$ iff

$$A_i\cdot A_j = A_j\cdot A_i \qquad (3.13)$$

for arbitrary $i,j,\in\{0,1,\ldots,k\}$. On the other hand the condition (3.9) can easily be written in the following form:

$$A_{s+1}\cdot A' = A_s\cdot A'' \qquad (3.14)$$

for $s\in\{0,1,\ldots,k-1\}$, where

$$A' = \begin{bmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0(m+k-1)} \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1(m+k-1)} \\ - & - & - & - \\ \alpha_{m0} & \alpha_{m1} & \cdots & \alpha_{m(m+k-1)} \end{bmatrix}$$

and

$$A'' = \begin{bmatrix} \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0(m+k)} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(m+k)} \\ - & - & - & - \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{m(m+k)} \end{bmatrix}.$$

Further, the equality (3.14) can be written in the following form:

$$A_{s+1} \cdot A_i = A_s \cdot A_{i+1} \tag{3.15}$$

for $i, s \in \{0, 1, \ldots, k-1\}$. Thus we only have to prove that the conditions (3.13) and (3.15) are equivalent.

It is easy to see that (3.13) is a consequence of (3.15), because applying the formula (3.15) $|j-i|$ times we obtain (3.13). In order to prove the converse we must use the fact that the matrices A_i are given by (3.12) and hence they are not mutually independent. We put $A_i = [x_i \; x_{i+1} \ldots x_{i+m}]$ $(i \in \{0, 1, \ldots, k\})$ where x_j $(j \in \{0, 1, \ldots, m+k\})$ is the following vector column

$$x_j = (\alpha_{0j}, \alpha_{1j}, \ldots, \alpha_{mj})^T.$$

The formulas (3.13) can now be written in the following form:

$$A_i \cdot x_{j+r} = A_j \cdot x_{i+r}$$

$i, j \in \{0, 1, \ldots, k\}$, $r \in \{0, 1, \ldots, m\}$, or

$$A_i \cdot x_p = A_j \cdot x_q$$

where $i, j \in \{0, 1, \ldots, k\}$, $p, q \in \{0, 1, \ldots, m+k\}$ and $i+p=j+q$. Hence we obtain

$$A_{s+1} \cdot x_i = A_s \cdot x_{i+1}$$

for $s \in \{0, 1, \ldots, k-1\}$, $i \in \{0, 1, \ldots, m+k-1\}$. This implies the equality (3.14) i.e. (3.15) and the proof of the theorem is finished. ∎

Similarly to Propositions 1.1 and 1.5, about the affine com(n,m)-semigroups, one can prove the following two propositions:

Proposition 3.4. If $(\mathbb{C}^*, f)$ is a projective com(m+1,m)- semigroup, then the induced com(m+k,m)-semigroup is also projective. ∎

Proposition 3.5. Let $(\mathbb{C}^*, f)$ be a com(m+1,m)-semigroup, such that for each $p \in \mathbb{C}^*$, $\bar{\varphi}_p$ is a linear transformation on $\mathbb{C}P^m$. Then $(\mathbb{C}^*, f)$ is a projective com(m+1,m)-semigroup. ∎

We notice that if we put $\alpha_{01}=\alpha_{02}=\ldots=\alpha_{0(m+k)}=0$ in the theorem 3.3, we obtain as a consequence the associativity law of the affine com(m+k,m)-groupoids. It can be verified that the condition of associativity (3.10) is invariant under the matrix transformations (3.4), (3.5) and (3.6). Now we will consider some examples of projective groups.

Example 3.1. Let us consider the projective com(3.2)- groupoid which is given by the following matrix:

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}.$$

It can be verified by Theorem 3.3 that the associative law is satisfied, and thus it determines a projective com(3,2)- semigroup. Its characteristic polynomial is

$$\Delta^*(z) = \begin{vmatrix} -z^3 & z^2 & -z & 1 \\ -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{vmatrix} = -(z-1)^2(z-2).$$

Thus, the only singular points are 1 and 2, and by removing them we obtain a projective com(3,2)-group on $\mathbb{C}^* \setminus \{1,2\}$. The singularity of the points 1 and 2 can directly be verified as follows. Let us consider the system

$$f(z_1, z_2, z) = (w_1, w_2)$$

where $z, w_1, w_2 \in \mathbb{C}$. By the change of variables $z_1+z_2=A$ and $z_1z_2=B$ this system is equivalent to the system

$$\begin{cases} (B+Az-5)/(A+z-4)=\alpha & (\alpha=w_1+w_2) \\ (Bz-2)/(A+z-4)=\beta & (\beta=w_1w_2) \end{cases} \tag{3.16}$$

i.e. to the system

$$\begin{cases} A(z-\alpha) + B = \alpha z - 4\alpha + 5 \\ A\beta - Bz = 4\beta - z\beta - 2 \end{cases} \tag{3.17}$$

Since the system (3.17) should have a unique solution for A and B, it should be

$$\begin{vmatrix} z-\alpha & 1 \\ \beta & -z \end{vmatrix} = -z^2 + z\alpha - \beta \neq 0 .$$

Let us suppose that z, α and β are chosen such that $z^2 - z\alpha + \beta = 0$. Then, $\beta = z(\alpha - z)$ and by replacing it in (3.14) one obtains the following system:

$$\begin{cases} A(z-\alpha) + B = \alpha(z-4) + 5 \\ Az(z-\alpha) + Bz = (z-4)z(\alpha-z)+2. \end{cases}$$

This system has infinitely many solutions iff $z^3-4z^2+5z-2=0$, i.e. $z_{1,2} = 1$ and $z_3 = 2$. Thus the points 1 and 2 are exactly the singular points. If $z\neq 1$ and $z\neq 2$ and $\beta=z(\alpha-z)$, then the system (3.17) does not have solutions in $\mathbb{C}$, but on the other hand the system (3.16) has the unique solution (A,B) in $\mathbb{C}P^2$ which is given by $A \to \infty$, $B \to \infty$ and $B/A=\alpha-p$. Thus there exists a unique pair $(z_1, z_2)\in(\mathbb{C}^*)^{(2)}$ such that $z_1+z_2=A$ and $z_1z_2=B$, namely $z_1=\infty$ and $z_2=\alpha-z$, or $z_1=\alpha-z$ and $z_2=\infty$. Similarly one can discuss the case when some of the elements z, w_1 and w_2 are ∞. Further we choose the following bilinear transformation $\Psi(z)=1/(z-1)$, such that $\varphi(1)=\infty$, and $\varphi(2)=1$. One can verify that this transformation induces an affine com(3,2)-group on $\mathbb{C}\setminus\{1\}$, and which is given by the following matrix:

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

In fact this is the affine com(3,2)-group from Example I.5.4. Its characteristic polynomial is

$$\Delta^*(z) = \begin{vmatrix} -z^3 & z^2 & -z & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = z-1$$

and hence the singular points are $z_{1,2}=\infty$ and $z_3=1$.

Example 3.2. Now we will consider a class of projective com(m+k,m)-groups, whose construction appears very naturally.

Let us denote by $PGL(m;\mathbb{C})$ the set of all non-singular linear transformations on $\mathbb{C}P^m$. In fact, $PGL(m;\mathbb{C})\cong GL(m+1;\mathbb{C})/\approx$ where $\approx$ is the equivalence relation on $GL(m+1;\mathbb{C})$ defined by

$$A \approx B \iff (\exists\lambda\in\mathbb{C}\setminus\{0\})\ A = \lambda B.$$

The group $PGL(m;\mathbb{C})$ is a complex Lie group and its complex dimension is $(m+1)^2-1=m(m+2)$.

Now let us suppose that A is an arbitrary $(m+1)x(m+1)$ complex matrix, and let

$$S(A) = S = \{\lambda\in\mathbb{C}^* \mid A+\lambda I\in PGL(m;\mathbb{C}) \text{ is a non-singular matrix}\},$$

i.e. $S=\mathbb{C}^*\setminus\{\lambda_1,\dots,\lambda_t\}$ where $\lambda_1,\dots,\lambda_t$ are the eigenvalues of the matrix A. Specially $\infty\in S$ because $A+\infty I \approx I$ is a non-singular matrix. Suppose that the minimal polynomial for A is

$$P(t) = t^{r+1} + a_r t^r + \dots + a_0.$$

It means that the matrices $I, A, \dots, A^r$ are linearly independent, and each of the matrices $A^{r+1}, A^{r+2}, \dots$ can be represented as a linear combination of $I, A, \dots, A^r$. This leads to a polynomial mapping from the set of all polynomials to the set of polynomials of degree $\le r$. Besides that, each of the polynomials of degree $\le r$ is a fixed point for that mapping. This mapping determines a com(r+k,r)-semigroup on $\mathbb{C}^*$ and com(r+k,r)-group on S. This com(r+k,r)-semigroup (group) is induced by com(r+1,r)-semigroup (group). Thus it is sufficient to study only such com(r+1,r)-semigroups (groups).

In the case of com(r+1,r)-semigroup (group), the polynomial

$$b_0 t^{r+1} + b_1 t^r + \dots + b_{r+1}$$

is mapped to

$$-b_0(a_r t^r + \dots + a_0) + b_1 t^r + \dots + b_{r+1}$$
$$= t^r(b_1-b_0a_r) + t^{r-1}(b_2-b_0a_{r-1}) + \dots + (b_{r+1}-b_0a_0).$$

Hence, this com(r+1,r)-semigroup (group) is projective, and it is given by the following matrix:

$$\begin{bmatrix} -a_r & 1 & 0 & 0 & \dots & 0 \\ -a_{r-1} & 0 & 1 & 0 & \dots & 0 \\ -a_{r-2} & 0 & 0 & 1 & \dots & 0 \\ \hline -a_0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{3.18}$$

It is also easy to check the associative law using Theorem 3.3. We notice that this projective com(r+1,r)-semigroup (group) depends only on the minimal polynomial $P(t)$ but not on the chosen matrix A. Herewith we obtain a natural class of projective groups which are parametrized by $r+1$ parameters. Now the following question arises. Whether each projective com(r+1,r)-group is isomorphic to any group of the above class of projective com(r+1,r)-groups? The answer is "no". Indeed each of these groups have the following property. The polynomial

$$0t^{r+1} + b_1t^r + \dots + b_{r+1}$$

of degree $r+1$ is mapped to the polynomial

$$b_1t^r + \dots + b_{r+1}$$

of degree r. Thus the first polynomial has roots ∞ and $z_1,\dots,z_r$, while the second has roots $z_1,\dots,z_r$. Hence there exists an element ξ ($\xi=\infty$ in this case) such that

$$f(\xi, z_1^r) = z_1^r$$

for each $z_1^r \in S^{(r)}$. Then the unit in the universal covering group is $(\xi,\dots,\xi)$ but this is not always the case (see the examples in I.5.). Conversely, let us suppose that there exists an element ξ in a projective com(r+1,r)-group such that

$$f(\xi, z_1^r) = z_1^r$$

for each $z_1^r \in S^{(r)}$. Then we choose a bilinear transformation Ψ such that $\Psi(\xi)=\infty$. According to Theorem 3.1 the induced com(r+1,r)-group is also projective and for it

$$f'(\infty, w_1^r) = w_1^r .$$

Hence each polynomial of degree $\le r$ is a fixed point for that polynomial mapping, and the corresponding matrix has the form (3.18).

The group G_h was introduced in II.2 in order to study the affine com(n,m)-group structures. Now in the projective case we shall introduce a similar group. In Example 3.2 above the group PGL(m; $\mathbb{C}$) was introduced.

Let a non-zero vector $h=h_0^m \in \mathbb{C}P^{m+1}$ be given. Let us denote by G_h the subset of PGL(m; $\mathbb{C}$) which consists of all the non-singular matrices

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0m} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m0} & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix} \qquad (3.19)$$

such that the vectors

$$(-\alpha_{01},\ \alpha_{00}-\alpha_{11},\ \alpha_{10}-\alpha_{21},\dots,\ \alpha_{(m-1)0}-\alpha_{m1},\ \alpha_{m0})$$
$$(-\alpha_{02},\ \alpha_{01}-\alpha_{12},\ \alpha_{11}-\alpha_{22},\dots,\ \alpha_{(m-1)1}-\alpha_{m2},\ \alpha_{m1})$$
$$- - - - - - - - - - - - - - - - - - -$$
$$(-\alpha_{0m},\alpha_{0(m-1)}-\alpha_{1m},\alpha_{1(m-1)}-\alpha_{2m},\dots,\alpha_{(m-1)(m-1)}-\alpha_{mm},\alpha_{m(m-1)})$$

are equal to h in $\mathbb{C}P^{m+1}$.

In the special case when $h_0=0$ all of the α_{0i} have to be 0 for $1\le i\le m$, i.e. $\alpha_{01}=\alpha_{02}=\dots=\alpha_{0m}=0$. Since the matrix (3.19) has to be non-singular it follows that $\alpha_{00}\neq 0$. But the matrix (3.19) belongs to $PGL(m;\mathbb{C})$, and thus it is determined up to a scalar multiple. If we put $\alpha_{00}=1$ then we obtain the group G_h defined in II.2 for $h=(h_1,h_2,\dots,h_{m+1})$. This gives a connection between the "affine" and "projective definition" of G_h.

Analogous to Theorem 2.1 one can prove the following theorem for the projective case:

Theorem 3.6. Let a vector $h\in\mathbb{C}P^{m+1}$ be given. The set of matrices G_h with the matrix multiplication is a closed Lie subgroup of $PGL(m;\mathbb{C})$ with complex dimension m. ■

In analogy with the proof of Theorem 2.1 one can verify that G_h coincides with the derived group $(G^{(m)},*)$ for arbitrary projective com(m+1,m)-group (G,f) whose characteristic polynomial is

$$h_0t^{m+1} - h_1t^m + h_2t^{m-1} - \dots +(-1)^{m+1}h_{m+1}.$$

The following theorems can be proved in the same way as Theorems 2.2, 1.11 and 2.3 for the affine case.

Theorem 3.7. Let a vector $h=h_0^m\in\mathbb{C}P^{m+1}$ be given, and let the polynomial

$$P(t) = h_0t^{m+1} - h_1t^m + h_2t^{m-1} - \dots +(-1)^{m+1}h_{m+1}$$

has exactly s different roots in $\mathbb{C}^*$ $(s\in\{1,\dots,m+1\})$. Then

$$G_h \cong \underbrace{C_1\times\dots\times C_1}_{s-1}\times\underbrace{C_0\times\dots\times C_0}_{m+1-s}. \quad ■ \tag{3.20}$$

Theorem 3.8. Let a non-singular projective com(m+k,m)- semigroup be given, and let the corresponding characteristic polynomial be $\Delta^*(z_1^k)$.

Then there exists a polynomial Δ'^* of one variable such that

$$\Delta^*(z_1^k) = \Delta'^*(z_1)\cdot\Delta'^*(z_2)\cdot \dots \cdot\Delta'^*(z_k) \tag{3.21}$$

and $\deg\Delta'^*(z)\le m+1$. ■

Theorem 3.9. Let a non-singular projective com(m+k,m)- semigroup on $\mathbb{C}^*$ (group on $\mathbb{C}^*\backslash\mathcal{R}$) be given, and suppose that its characteristic polynomial has exactly s different roots in $\mathbb{C}^*$. Then the given projective com(m+k,m)-semigroup (group) is induced by k^{s-1} different projective com(m+1,m)-semigroups on $\mathbb{C}^*$ (groups on $\mathbb{C}^*\backslash\mathcal{R}$). ■

In this chapter we have not considered the question about isomorphism between two projective (affine) com(m+k,m)- groups (semigroups). Having in mind Theorem 3, the following question appears naturally. If two com(m+k,m)-groups (semigroups) are isomorphic, is the isomorphism induced by a bilinear transformation? If we know that the answer of this question is affirmative, then using compositions of the matrix transformations (3.4), (3.5) and (3.6) we would be able to deduce whether two com(m+k,m)- groups (semigroups) are isomorphic or not.