Математички Билтен
17 (XLIII)
1993 (93-100)
Скопје, Македонија

ISSN 0351-336X

ITERATIVE FORMULAS FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS OF I AND II ORDER

M.Kujumdzieva-Nikoloska

Abstract. In this paper some iterative methods (1.2)((1.3)), (2.2)((2.3)),(3.3)((3.4)) for solving homogenous linear differential equations (1), (2), (3) of I and II order with analytical coefficients are presented.

I. For the linear homogenous differential equation

$$y' + a(x)y = 0 \tag{1}$$

with the initial conditions $y_0=y(0)=c_1$, presuming that $a(x)$ is analytical, on $D = \{(x,y) \mid |x| \le x_0 \le \alpha, \ |y-y_0| = |y-c_1| \le \beta\}$, the following holds:

Theorem. If

1^0 $|a(x)| < M$ for $|x| \le \alpha$, ($a(x)$ is analytical),

2^0 $|y(x)| \le \beta + |c_1| = Y$ for $|x| \le \alpha$, ($y(x)$ is analytical)

3^0 $h \le \min(\alpha, \frac{\beta}{MY})$,

then there exists unique solution of the Cauchy's problem (1) in $I = \{x \mid |x| \le h\}$.

Proof. From $y' = -a(x)y$ it follows that

$$\int_{y_0}^{y} dy = -\int_0^x a(x)ydx$$

$$y - y_0 = -\int_0^x a(x)ydx$$

i.e.

$$y = c_1 - \int_0^x a(x)ydx \tag{1.1}$$

On the closed part E of the space C(I), (H(I)) for which $||y-c_1|| = \max_I |y-c_1| \le \beta$, we define the mapping

$$Ty = c_1 - \int_0^x a(x)ydx \quad (y \in E)$$

Observe that this mapping is continuous. We shall prove now, that T maps E into E.

Let $x\in I$, $y\in E\subset C(I)$. Then $Ty\in C(I)$, since

$$|Ty-c_1| \leq |\int_0^x a(x)y\,dx| \leq \int_0^{x_0} |a(x)y|\,dx \leq Mx_0Y \leq MY\alpha \leq \beta \text{ for } \alpha \leq \frac{\beta}{MY},$$

i.e. from $|y-c_1| \leq \beta$ it follows that $|Ty-c_1| \leq \beta$, and $Ty\in E$.

Further, since

$$|Ty_1-Ty_2| = |\int_0^x a(x)(y_1-y_2)dx| \leq M \max_{x\in I}|y_1-y_2||x| \leq M \max_{x\in I}|y_1-y_2|h,$$

$$\begin{aligned}|T^2y_1-T^2y_2| &= |T(Ty_1)-T(Ty_2)| \leq |\int_0^x a(x)(Ty_1-Ty_2)dx| \leq \\ &\leq |\int_0^x a(x)(\int_0^x a(x)(y_1-y_2)dx)dx| \leq M^2\max_{x\in I}|y_1-y_2|\frac{x^2}{2!} \leq \\ &\leq M^2\max_{x\in I}|y_1-y_2|\frac{h^2}{2!}\end{aligned}$$

and so on,

$$\begin{aligned}|T^ny_1-T^ny_2| &\leq |T(T^{n-1}y_1)| - T(T^{n-1}y_2)| \leq \\ &\leq |\int_0^x a(x)(T^{n-1}y_1-T^{n-1}y_2)dx| \leq \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq |\int_0^x a(x)(\int_0^x a(x)(\ldots(\int_0^x a(x)(y_1-y_2)dx)dx\ldots)dx| \leq \\ &\leq M^n\max|y_1-y_2||\int_0^x dx\int_0^x dx\ldots\int_0^x dx| \leq \\ &\leq M^n\max|y_1-y_2|\frac{h^n}{n!}\end{aligned}$$

Now, we can choose n to be big enough, so that

$$\frac{M^nh^n}{n!} < 1,$$

and then

$$||T^ny_1-T^ny_2|| < ||y_1-y_2||$$

which proves that T is a contraction, and then, ([2],[3]), the equation

$$y = Ty$$

has one and only one solution. This solution is the limit y of the iterative sequence $\{y_n\}$, defined by

$$y_{n+1} = Ty_n, \quad n=0,1,2,\ldots$$

$$y_{n+1} = c_1 - \int_0^x a(x)y_n dx \qquad (1$$

i.e.

$$y_{n+1} = c_1[1-\int_0^x a(x)dx + \int_0^x a(x)dx\int_0^x a(x)dx... +$$
$$+ (-1)^n \int_0^x a(x)dx\int_0^x a(x)dx...\int_0^x a(x)dx] =$$
$$= c\ (1+\sum_{k=1}^{n} (-1)^k \underbrace{\int_0^x a(x)dx\int_0^x a(x)dx...\int_0^x a(x)dx}_{k}) \qquad (1.3)$$

II. For the differential equation

$$y'' + a(x)y = 0 \qquad (2)$$

with the initial conditions $y_0 = y(0) = c_1$, $y_0'=y'(0)=c_2$, on $D = \{(x,y) \mid |x| \le x_0 \le \alpha,\ |y-y_0-y_0'x| \le \beta\}$ the following holds:

Theorem. Let

1^o $|a(x)| \le M$ for $|x| \le \alpha$, ($a(x)$ is analytic),

2^o $|y(x)| \le \beta + |c_1+c_2x| = Y$ for $|x| \le \alpha$, ($y(x)$ is analytical)

3^o $h \le \min(\alpha, \sqrt{\frac{2\beta}{MY}})$.

than the Cauchy's problem (2) has unique solution in $I = \{x \mid |x| \le h\}$.

Proof. From $y'' = -a(x)y$ it follows that

$$\int_0^x dy' = -\int_0^x a(x)ydx,$$
$$y'\Big|_{y_0}^{y} = -\int_0^x a(x)ydx$$
$$y' = y_0' - \int_0^x a(x)ydx,$$
$$\int_{y_0}^{y} dy = \int_0^x y_0'dx - \int_0^x(\int_0^x a(x)ydx)dx$$
$$y - y_0 = y_0'x - \int_0^x\int_0^x a(x)ydx^2$$

i.e.

$$y = c_1 + c_2x - \int_0^x\int_0^x a(x)ydx^2 \qquad (2.1)$$

On the closed part E of the space C(I), (A(I)) for which $\max_I |y-c_1-c_2x| \le \beta$, we define a mapping T by

$$Ty = c_1 + c_2x - \int_0^x\int_0^x a(x)ydx^2, \quad (y\in E),$$

As in the first Theorem (part I), we observe that T is continuous, and we shall prove that T maps E into E.

Let $x \in I$, $y \in E$ $(\subset C(I))$, than obviously $Ty \in E \subset C(I)$, i.e. we have that

$$|Ty-c_1-c_2x| = \left|\int_0^x\int_0^x a(x)y\,dx\right| \le MY\left|\int_0^x\int_0^x dx^2\right| \le MY\frac{x_0^2}{2!} \le MY\frac{\alpha^2}{2} \le \beta, \text{ for}$$

$\alpha \le \sqrt{\frac{2\beta}{MY}}$, which proves that $|y-c_1-c_2x| \le \beta$ implies $|Ty-c_1-c_2x| \le \beta$, i.e. $Ty \in E$.

Now, let us prove that T^n is a contraction.

Let $y_1, y_2 \in E$, then, for $x \in I$, we have that

$$|Ty_1-Ty_2| \le \left|\int_0^x\int_0^x a(x)(y_1-y_2)\,dx\right| \le M\max_I|y_1-y_2|\frac{x^2}{2!} \le M\max_I|y_1-y_2|\frac{h^2}{2!}$$

Further

$$|T^2y_1-T^2y_2| \le |T(Ty_1)-T(Ty_2)| \le \left|\int_0^x\int_0^x a(x)(Ty_1-Ty_2)\,dx^2\right| =$$

$$= \left|\int_0^x\int_0^x a(x)\,dx^2\int_0^x\int_0^x a(x)(y_1-y_2)\,dx^2\right| \le M^2\max_I|y_1-y_2|\frac{x^4}{4!} \le$$

$$\le M\max_I|y_1-y_2|\frac{h^4}{4!}$$

and so on,

$$|T^ny_1-T^ny_2| \le |T(T^{n-1}y_1)-T(T^{n-1}y_2)| \le \left|\int_0^x\int_0^x a(x)(T^{n-1}y_1-T^{n-1}y_2)\,dx^2\right| \le$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\le \left|\int_0^x\int_0^x a(x)\,dx^2\int_0^x\int_0^x a(x)\,dx^2\ldots\int_0^x\int_0^x a(x)(y_1-y_2)\,dx^2\right| \le$$

$$\le M^n\max_I|y_1-y_2|\frac{x^{2n}}{(2n)!!} \quad \frac{M^nh^{2n}}{(2n)!!}\max_I|y_1-y_2|$$

By taking maximum (over I) on the left and right side, we get that

$$||T^ny_1-T^ny_2|| \le \frac{M^nh^{2n}}{(2n)!}||y_1-y_2||$$

since for n big enough, we have that $\frac{M^nh^{2n}}{(2n)!} < 1$ it follows that T^n is a contraction, i.e. ([2],[3]), T is a contraction. This implies that the equation

$$y = Ty$$

has one and only one solution which can be obtained as a limit the iterative sequence $\{y_n\}$ defined by

$$y_{n+1} = Ty_{n}, \quad n=0,1,2,\dots$$

$$y_{n+1} = c_1+c_2x - \int_0^x\int_0^x a(x)y_n dx^2 \tag{2.2}$$

$$\begin{aligned} y_{n+1} &= c_1\{1-\int_0^x\int_0^x a(x)dx^2 + \dots + (-1)^n\int_0^x\int_0^x a(x)dx^2 \dots \int_0^x\int_0^x a(x)dx^2\} + \\ &+ c_2\{x-\int_0^x\int_0^x xa(x)dx^2 + \dots + (-1)^n\int_0^x\int_0^x a(x)dx^2 \dots \int_0^x\int_0^x xa(x)dx^2\} = \\ &= c_1\{1+\sum_{k=1}^{n}(-1)^k\underbrace{\int_0^x\int_0^x a(x)dx^2 \dots \int_0^x\int_0^x a(x)dx^2}_{k}\}^{n} + \\ &+ c_2\{x+\sum_{k=1}^{n}(-1)^k\underbrace{\int_0^x\int_0^x a(x)dx^2 \dots \int_0^x\int_0^x xa(x)dx^2}_{k}\} \end{aligned} \tag{2.3}$$

III. For the linear homogenous differential equation

$$y'' + a(x)y' + b(x)y = 0 \tag{3}$$

with the initial conditions $y_0=y(0)=c_1$ and $y_0'=y'(0)=C_2$, presuming that $a(x)$ and $b(x)$ are analytical, on $D = \{(x,y) \mid |x| \le x_0 \le \alpha, |y-c_1-c_2x| \le \beta, c_2=C_2+c_1a(0)\}$, the following theorem is true:

<u>Theorem</u>. Let

$\underline{1}^{\circ}$ $|a(x)| \le M$, $|a'(x)| \le M_1$, $|b(x)| \le B$, and $M^*=\max\{M,B+M_1\}$, ($a(x)$, $b(x)$ are analytical),

$\underline{2}^{\circ}$ $|y(x)| \le \beta + |c_1+c_2x| = Y$ for $|x| \le \alpha$, ($y(x)$ is analytical),

$\underline{3}^{\circ}$ $h \le \min(\alpha, \sqrt{1+\frac{2\beta}{M^*Y}}-1)$.

Then in $I = \{x \mid |x| \le h\}$ there exists a unique solution of the Cauchy's problem (3).

<u>Proof</u>. From

$$y'' = -a(x)y' - b(x)y = -(a(x)y)' - (b(x)-a'(x))y$$

it follows that $\int_{y_0}^{y} dy' = -\int_0^x a(x)y'dx - \int_0^x b(x)ydx$

$$y' = y_0' - \int_0^x a(x)y'dx - \int_0^x b(x)ydx$$

$$\int_{y_0}^{y} dy = \int_0^x y'dx - \int_0^x dx(\int_0^x a(x)y'dx) - \int_0^x dx\int_0^x b(x)ydx$$

$$y = y_0 + y_0'x - \int_0^x dx\int_0^x a(x)y'dx - \int_0^x dx\int_0^x b(x)ydx$$

i.e.

$$y = c_1+C_2x - \int_0^x dx\int_0^x a(x)y'dx - \int_0^x dx\int_0^x b(x)ydx \tag{3.1}$$

By using the method of partial integration in the second part of the above formula, we get that

$$y = c_1+c_2x - \int_0^x a(x)ydx - \int_0^x dx\int_0^x (b-a')ydx \tag{3.2}$$

In the closed part E of the spaçe C(I) (A(I)) for which $\max_{x\in I}|y(x)-c_1-c_2x| \le \beta$, we define the mapping T by

$$Ty = c_1+c_2x - \int_0^x a(x)ydx - \int_0^x dx\int_0^x (b(x)-a'(x))ydx$$

for$\in y\in E$. The mapping T is continuous, we shall prove now, that T maps E into E.

Let $x\in I$, $y\in E(\subset C(I))$. Then obviously $Ty\in E\subset C(I)$, i.e. we have that

$$|Ty-c_1-c_2x| \le |\int_0^x a(x)ydx| + |\int_0^x dx\int_0^x (b-a')ydx| \le$$

$$\le MY\alpha + (B+M_1)Y\frac{\alpha^2}{2} \le M^*Y(\frac{\alpha^2}{2}+\alpha) \le \frac{M^*Y}{2}(\alpha+2)\alpha \le \beta$$

$$(\le M^*Y(\alpha^2+\alpha) \le M^*Y\alpha(\alpha+1) \le \beta)$$

where

$$\alpha \le \sqrt{1+\frac{2\beta}{M^*Y}} - 1 \qquad (\text{or } \alpha \le \sqrt{\frac{1}{4}+\frac{2\beta}{M^*Y}} - \frac{1}{2})$$

From $|y-c_1-c_2x| \le \beta$, it follows that $|Ty-c_1-c_2x| \le \beta$, i.e. $Ty\in E$.

Let us prove that T^n is a contraction.

Let $y_1,y_2\in E$, then for $x\in I$ we have that

$$|Ty_1-Ty_2| = |\int_0^x a(x)(y_1-y_2)dx + \int_0^x\int_0^x (b-a')(y_1-y_2)dx| \le$$

$$\le M^*|y_1-y_2|(|\int_0^x dx| + |\int_0^x\int_0^x dx^2|) \le M^*|y_1-y_2|(\frac{|x|}{1}+\frac{x^2}{2!}) \le$$

$$\le M^*|y_1-y_2|\frac{|x|(h+1)}{1!}$$

$$|T^2y_1-T^2y_2| = |T(Ty_1)-T(y_2)| =$$

$$= |\int_0^x a(T(y_1-Ty_2)dx + \int_0^x\int_0^x (b-a')(Ty_1-Ty_2)dx^2| \le$$

$$\le M^{*2}|y_1-y_2|\left(\frac{|x|+1x^2}{2!} + \frac{(|x|+1)|x|^3}{3!}\right) \le$$

$$\le M^{*2}\frac{|y_1-y_2|}{2!}x^2(h+1)(1+h) \le$$

$$\le M^{*2}\frac{|y_1-y_2|}{}x^2(1+h)^2$$

Now, let $|T^n y_1 - T^n y_2| \le \frac{M^{*n}|x|^n(1+h)^n}{n!}|y_1-y_2|$. Then

$$|T^{n+1}y_1 - T^{n+1}y_2| = |T(T^n y_1) - T(T^n y_2)| \le$$

$$\le |\int_0^x a(T^n y_1 - T^n y_2)dx + \iint_{00}^{xx}(b-a')(T^n y_1 - T^n y_2)dx^2| \le$$

$$\le M^* M^{*n}|y_1-y_2|\left(\frac{|x|^{n+1}}{(n+1)!}(1+h)^n + \frac{(1+h)^n|x|^{n+2}}{(n+2)!}\right) \le$$

$$\le M^{*n+1}|y_1-y_2|\frac{|x|^{n+1}}{(n+1)!}(1+h)^n(1+h) \le$$

$$\le M^{*n+1}|y_1-y_2|\frac{h^{n+1}(1+h)^{n+1}}{(n+1)!} = \frac{(M^*h(1+h))^{n+1}}{(n+1)!}|y_1-y_2|$$

So, we get that

$$\max|T^{n+1}y_1 - T^{n+1}y_2| \le \frac{(M^*h(1+h))^{n+1}}{(n+1)!}\max_I|y_1-y_2|$$

or

$$||T^{n+1}y_1 - T^{n+1}y_2|| \le \frac{(M^*h(1+h))^{n+1}}{(n+1)!}||y_1-y_2||$$

Since, for n large enough, we have that

$$\frac{(M^*h(1+h))^n}{n!} < 1$$

it follows that T^n is a contraction, and then the equation

$$y = Ty$$

has a unique solution in I which can be obtained as limit of the sequence $\{y_n\}$ defined by

$$y_{n+1} = Ty_n, \quad n=0,1,2,\ldots \text{ , i.e.}$$

i.e. $y_{n+1} = c_1+c_2x - \int_0^x a(x)y_n dx - \int_0^x dx\int_0^x (b-a')y_n dx$, $(n=0,1,2,\ldots)$ (3.3)

and

$$y_0 = c_1 + c_2 x$$

$$y_1 = c_1[1-\int_0^x adx - \iint_{00}^{xx}(b-a')dx^2] + c_2[x-\int_0^x xadx - \iint_{00}^{xx}(b-a')xdx^2]$$

$$y_2 = c_1[1-\int_0^x adx+\int_0^x adx\int_0^x adx-\iint_{00}^{xx}(b-a')dx^2+\int_0^x adx\iint_{00}^{xx}(b-a')dx^2 +$$
$$+ \iint_{00}^{xx}(b-a')dx^2\int_0^x adx+\iint_{00}^{xx}(b-a')dx^2\iint_{00}^{xx}(b-a')dx^2] +$$
$$+ c_2[x-\int_0^x axdx+\int_0^x adx\int_0^x axdx-\iint_{00}^{xx}(b-a')xdx^2 +$$
$$+ \int_0^x adx\iint_{00}^{xx}(b-a')xdx^2+\iint_{00}^{xx}(b-a')dx^2\int_0^x xadx+\iint_{00}^{xx}(b-a')dx^2\iint_{00}^{xx}(b-a'xdx^2]$$

e.t.c.

Remark 1. The above considerations holds also when $a(x)\in C(I)$, or $a(x)\in C^1(I)$, $b(x)\in C(I)$.

Remark 2. The case I can be considered a special case of the standard iterative method ([1], [2], [3],...). This way of aproximate solwing is attractive for differential equation of III and higher order and also for systems of differential equations.

We have obtained adequate iterative processes more efficient from already known methods. They are subject of separate work.

R E F E R E N C E S

[1] Kolatz, L.: Funkcionalnij analiz i vijcislitelnaja matematika, Moskva, 1969

[2] Krasnoselskij, M.A.: Priblizennoe resenie operatornij uravnenij, Moskva, 1969

[3] Milanović, G.: Numerička analiza, II deo, Beograd, 1988

ИТЕРАТИВНИ ФОРМУЛИ ЗА РЕШАВАЊЕ НА ЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ ОД I И II РЕД

Марија Кујумџиева-Николоска

Р е з и м е

Во овој труд се дадени итеративни методи (1.2)((1.3)), (2.2)((2.3)),(3.3)((3.4)) за решавање на линеарните диференцијални равенки (1), (2), (3) од I и II ред со аналитички коефициенти. Истите методи важат и ако $a(x)\in C(I)$, односно $a(x)\in C^1(I)$, и $b(x)\in C(I)$.