ON SEQUENCES ASSOCIATED WITH FREE (1,2)-GROUPOIDS

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Abstract

General information on vector valued groupoids and $(n,m)-$ operations can be found in [1]. In this paper we define and consider $(1,2)-$sequences, which are related to $(1,2)-$operations. By using these $(1,2)-$sequences we give a combinatorial proof of a problem posed in [2] (different to that given in [1]) that every free $(1,2)-$groupoid is a proper $(1,2)-$subsemigroup of a semigroup. Several combinatorial properties of $(1,2)-$sequences are also obtained.

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1. $(1,2)-$ operations

Let Q be a nonempty set, and denote by $Q^n (n \geq 0)$ the $n-$th cartezian power of Q. A mapping $h : Q^n \to Q^m (n, m \geq 1)$ is said to be an $(n,m)-$operation on Q, and in the case $n = 1$, $m = 2$ we have a $(1,2)-$operation on Q. Then the pair $(Q; h)$ is said to be a $(1,2)-$grupoid. Note that any $(1,2)-$operation can be considered as a pair $h = (h_1, h_2)$ of two unary operations h_1 and h_2 on Q, defined by

$$h(x) = (y, z) \iff h_1(x) = y, \;\; h_2(x) = z.$$

Let h be a $(1,2)-$operation on a set Q, and let t, s be nonnegative integers. By $\mathcal{P}(h; p, s)$ (or, shortly, $\mathcal{P}(p, s)$) we denote the set of all polynomial $(1 + p, 1 + p + s)-$ operations on Q, obtained from h as follows:

Let $i_1, ..., i_s, j_1, ..., j_s$ be nonnegative integers, such that

$$i_{n+1} + j_{n+1} = p + n, \;\; n = 0, 1, ..., s - 1,$$

and denote by 1 the identity mapping on Q. Then $\mathcal{P}(p, s)$ consists of all $(1 + p, 1 + p + s)-$ operations of this form:

$$(1^{i_s} \times h \times 1^{j_s}) \cdot ... \cdot (1^{i_2} \times h \times 1^{j_2}) \cdot (1^{i_1} \times h \times 1^{j_1}),$$

where by " $\cdot$ " is denoted the usual composition of mappings, and by " $\times$ " is denoted the cross-product of mappings (i.e. if $h : Q^n \to Q^m$, $g : Q^k \to Q^r$, then $h \times g : Q^{n+k} \to Q^{m+r}$ is defined by

$$(h \times g)(x_1, ..., x_n, y_1, ..., y_k = (h(x_1, ..., x_n), g(y_1, ..., y_k))).$$

Here, $1^k = \underbrace{1 \times ... \times 1}_{k}$ is the identity mapping on Q^k, and we take $\mathcal{P}(p, 0) = \{1^{1+p}\}$.

It is clear that the cardinality of the set $\mathcal{P}(p, s)$ is $(p + s)!/p!$.

Further on, instead of $\mathcal{P}(0, s)$, we write $\mathcal{P}(s)$. Thus we have:

$$\mathcal{P}(0) = \{1\}, \;\; \mathcal{P}(1) = \{h\}, \;\; \mathcal{P}(2) = \{(h \times 1)h, \; (1 \times h)h\},$$

$$\mathcal{P}(3) = \{(h \times 1^2)(h \times 1)h, \;\; (1 \times h \times 1)(h \times 1)h, \; (1^2 \times h)(h \times 1)h,$$
$$(h \times 1^2)(1 \times h)h, \; (1 \times h \times 1)(1 \times h)h, \; (1^2 \times h)(1 \times h)h\}, ...$$

2. $(1,2)-$subsemigroups of semigroups

A $(1,2)-$groupoid $\mathcal{Q} = (Q; h)$ is said to be a $(1,2)-$subsemigroup of a semigroup $\mathcal{S} = (S; \cdot)$ iff $Q \subseteq S$ and, for every $a, b, c \in Q$, the following implication holds:

$$h(a) = (b, c) \Rightarrow a = b \cdot c.$$

Moreover, if the equivalence

$$h(a) = (b, c) \Longleftrightarrow a = b \cdot c$$

holds in $\mathcal{Q}$, then we say that Q is a proper $(1,2)-$subsemigroup of the semigroup $\mathcal{S}$.

Next Theorem was proved in [1]:

Theorem 2.1. *a) A $(1,2)-$groupoid $Q = (Q; h)$ is a $(1,2)-$subsemigroup of a semigroup iff for each $s \geq 0$ and every $t, g \in \mathcal{P}(s)$, the implication*

$$t(x) = g(y) \Rightarrow x = y \tag{2.1}$$

is true in $\mathcal{Q}$ for every $x, y \in Q$.

b) $\mathcal{Q}$ is a proper $(1,2)-$subsemigroup of a semigroup iff, besides (2.1), for each $s \geq 0$ and every $t \in \mathcal{P}(s)$, $g \in \mathcal{P}(1, s)$, the implication

$$t(x) = g(y_1, y_2) \Rightarrow h(x) = (y_1, y_2) \tag{2.2}$$

is true in $\mathcal{Q}$ for every $x, y_1, y_2 \in Q$. □

Let B be a nonempty set, and define a set FB inductively, as follows:

$$B_0 = B,\ B_{p+1} = B_p \cup \{(u, i) \mid u \in B_p,\ i \in \{1, 2\}\} \text{ for } p \geq 0, \text{ and}$$

$$FB = \cup(B_p \mid p \geq 0).$$

Define a $(1,2)-$operation f on FB by

$$f(u) = ((u, 1), (u, 2))$$

for each $u \in FB$. Then $\mathcal{FB} = (FB; f)$ is a free $(1,2)-$groupoid, freely generated by the set B, according to the following property:

Theorem 2.2. *Let $\mathcal{Q} = (Q; h)$ be a $(1,2)-$groupoid and $\varphi : B \to Q$ a mapping. Then φ can be uniquely extended to a homomorphism $\phi^* : \mathcal{FB} \to \mathcal{Q}$, i.e. such that*

$$\varphi^*(f_i(u)) = h_i(\varphi^*(u)), \;\; i = 1, 2, \tag{2.3}$$

where $f = (f_1, f_2)$, $h = (h_1, h_2)$.

Proof. Let $\varphi_0 = \varphi : B_0 \to Q$, and suppose that $\varphi_p : B \to Q$ is defined. Then define φ_{p+1} by

$$\varphi_{p+1} = \begin{cases} \varphi_p(u), & \text{if } u \in B_p \\ h_i(\varphi_p(v)), & \text{if } u = (v, i) \in B_{p+1}B_p, i = 1, 2, \end{cases}$$

and take $\varphi^* = \cup(\varphi_p \mid p \geq 0)$ to be the union of the chain of mappings $\varphi_0 \subseteq \varphi_1 \subseteq \varphi_2 \subseteq ...$ Clearly, (2.3) is satisfied and φ^* is unique, since B is a generating subset of $\mathcal{FB}$. □

In what follows we give another proof of the property that $\mathcal{FB}$ is a proper $(1,2)-$subsemigroup of a semigroup [1]. For this aim we use $(1,2)-$sequences of integers, defined below. In our opinion, these $(1,2)-$sequences are self-interesting as well.

3. $(1,2)-$sequences

At first, we make a modification in the denotation of the elements of a free $(1,2)-$groupoid $\mathcal{FB}$. Namely, by the construction of the elements of FB, we can take that the elements of $B_{p+1}(p \geq 0)$ have the form (b, a_1^p), where $b \in B$, $a_1^p = a_1 a_2 ... a_p$, $a_j \in \{1, 2\}$ for $j = 1, 2, ..., p$; if $p = 0$, we take that (b, a_1^0) denotes simply the element $b \in B$.

Thus, for the $(1,2)-$ operation f on FB we have

$$f(u, a_1^p) = ((u, a_1^{p_1}, (u, a_1^{p_2})), \;\; \text{i.e. } f_i(u, a_1^p) = (u, a_1^{p_i}), \; i = 1, 2.$$

As before, by $\mathcal{P}(p, s)$ we denote the set of all $(1+p, 1+p+s)-$ operations on $\mathcal{FB}$, obtained by the $(1,2)-$ operation f. If $u = (b, a_1^r) \in FB$ then we say that r is the depth of u and we denote $d(u) = r$. If $g \in \mathcal{P}(s)$, $s \geq 0$, $u \in FB$, $d(u) = r$ and

$$g(u) = u_0 u_1 ... u_s, \ u_j \in FB, \ j = 0, 1, ..., s$$

then $d(u_j) = r + a_j$, $j = 0, 1, ..., s$, where a_j are nonnegative integers. Note that a_j are positive integers if $s > 0$, and $u_0 = u$, $a_0 = 0$ if $s = 0$. It is clear that the integers a_j do not depend on the element $u \in FB$. In such a way, to every element $g \in \mathcal{P}(s)$ we correspond a unique sequence of integers $\lambda(g) = a_0 a_1 ... a_s$. We say that $\lambda(g)$ is the $(1,2)-$ sequence of g of rank s, and $\Lambda = \cup\{\lambda(g) \mid g \in \mathcal{P}(s)\}$ is called the set of $(1,2)-$ sequences. Thus we have:

s	$g \in \mathcal{P}(s)$	$\lambda(g)$
0	1	0
1	f	11
	$(f \times 1)f$	221
2	$(1 \times f)f$	122
	$(f \times 1^2)(f \times 1)f$	3321
	$(1 \times f \times 1)(f \times 1)f$	2331
	$(1^2 \times f)(f \times 1)f$	2222
	$(f \times 1^2)(1 \times f)f$	2222
	$(1 \times f \times 1)(1 \times f)f$	1332
3	$(1^2 \times f)(1 \times f)f$	1233

We note that the elements of Λ can be generated from O by using this rule:

(3.1) $a_0 ... a_s \in \Lambda \Rightarrow a_0 ... a_{i-1}(a_i + 1)(a_i + 1)a_{i+1} ... a_s \in \Lambda, \ i = 0, 1, ..., s.$

Thus, every element $\sigma \in \Lambda$ of rank s derives $s+1$ elements of Λ of rank $s+1$ (but different elements of Λ of rank s can derive a same element of Λ of rank $s+1$). Also, every of Λ of rank $s > 0$ is derived from 0 by using the rule (3.1) s times.

Let $\sigma = a_0 a_1 ... a_n \in \Lambda$ be with a rank n, and let k be an integer. We will use these functions:

$x_\sigma(k)-$ the number of appearences of k in σ,

$m_\sigma = \max_{0 \leq i \leq n} a_i$,

$s_\sigma(r) = \sum_{k=0}^{r} x_\sigma(k), \ r \geq 0,$

$y_\sigma(k)$– the number of replacements of $k-1$ with kk in a derivation of σ from 0 by using rule (3.1). (Evidently, $y_\sigma(k)$ depends only of σ and k).

The preceding functions have these (eassily checked) properties:

$$m_\sigma \leq n, \tag{3.2}$$

$$s_\sigma(m_\sigma) = n+1, \tag{3.3}$$

$$k \geq 2 \Rightarrow x_\sigma(k) = s_\sigma(k) - s_\sigma(k-1),\ x_\sigma(i) = s_\sigma(i) \text{ for } i = 0,1, \tag{3.4}$$

$$y_\sigma(0) = y_0(k) = 0,\ k \geq m_\sigma \Rightarrow y_\sigma(k) = 0, \tag{3.5}$$
$$\sigma \neq 0 \Rightarrow y_\sigma(1) = 1 \text{ and } y_\sigma(k) \geq 1 \text{ for } k = 2,3,...,m_\sigma,$$

$$y_\sigma(m_\sigma) = \frac{x_\sigma(m_\sigma)}{2},\ y_\sigma(k) = \frac{x_\sigma(k) + y_\sigma(k+1)}{2} \tag{3.6}$$
$$\text{for } \sigma \neq 0 \text{ and } k = 1,...,m_\sigma - 1,$$

$$\sum_{k=0}^{n} y_\sigma(k) = \sum_{k=0}^{m_\sigma} y_\sigma(k) = n. \tag{3.7}$$

We will prove this proposition:

Theorem 3.1. *Let $\xi = a_0a_1...a_n$, $\psi = b_0b_1...b_n \in \Lambda$, $n \geq 0$. Then there exist $i, j \in \{0,1,...,n\}$ such that $a_i \leq b_i$ and $a_j \leq b_j$.*

Proof. The property is clear if $m = m_\xi = m_\psi$. Assume that $m = m_\xi > m_\psi$. Then there is some j such that $a_j = m > m_\psi \geq b_j$, and therefore we have to show that $a_i \leq b_i$ for some i.

A top-down recursion applied to (3.6) gives:

$$y_\xi(m) = \frac{x_\xi(m)}{2},$$

$$y_\xi(m-1) = \frac{2x_\xi(m-1) + x_\xi(m)}{2^2}, ...,$$

$$y_\xi(k) = \frac{2^{m-k}x_\xi(k) + ... + 2x_\xi(m-1) + x_\xi(m)}{2^{m-k+1}}, ...,$$

$$y_\xi(1) = \frac{2^{m-1}x_\xi(1) + ... + 2x_\xi(m-1) + x_\xi(m)}{2^m},\ y_\xi(0) = 0.$$

In the same manner, we have

$$y_\psi(k) = \frac{2^{m-k}x_\psi(k) + ... + 2x_\psi(m-1) + x_\psi(m)}{2^{m-k+1}}, \quad k = 1, 2, ..., m.$$

Now, (3.7) implies:

$$\sum_{k=1}^{m}(\sum_{i=1}^{k} 2^{m-i})x_\xi(k) = \sum_{k=1}^{m}(\sum_{i=1}^{k} 2^{m-i})x_\psi(k), \tag{3.8}$$

and, applying (3.4) to (3.8), we have

$$\sum_{k=1}^{m-1} 2^{m-k-1}s_\xi(k) = \sum_{k=1}^{m-1} 2^{m-k-1}s_\psi(k). \tag{3.9}$$

We note that $x_\xi(m) > x_\psi(m) = 0$ implies $s_\xi(m-1) < s_\psi(m-1) = n+1$, i.e.

$$\sum_{k=1}^{m-2} 2^{m-k-1}s_\xi(k) > \sum_{k=1}^{m-2} 2^{m-k-1}s_\psi(k). \tag{3.9'}$$

Hence, there is a $k \in \{1, 2, ..., m-2\}$ such that $s_\xi(k) > s_\psi(k)$, i.e.

$$x_\xi(1) + ... + x_\xi(k) > x_\psi(1) + ... + x_\psi(k). \tag{3.10}$$

The inequality (3.10) means that the sequence ξ contains more members from the set $\{1, 2, ..., k\}$ than the sequence ψ, and so there exists an index i $(0 \leq i \leq n)$ such that $a_i \in \{1, 2, ..., k\}$, $b_i \notin \{1, 2, ..., k\}$.

Thus we obtained that $a_i < b_i$. □

Consider now a $(1,2)-$ sequence $\xi = a_0a_1...a_p \in \Lambda$ $(p \geq 0)$ and let α be an integer. If $a_i + \alpha \geq 1$ for each $i \in \{0, 1, 2, ..., p\}$, then we denote by $[\xi + \alpha] = (a_0 + \alpha)(a_1 + \alpha)...(a_p + \alpha)$ the sequence of integers obtained from ξ by adding an α to each of its members a_i.

The following proposition is true:

Theorem 3.2. *Assume that the integers* α, β *and the* $(1,2)-$*sequences* $\xi = a_0a_1...a_p$, $\psi = b_0b_1...b_q$ *are such that* $a_p + \alpha \geq 1$, $b_q + \beta \geq 1$ *for each* $i = 0, 1, ..., p$, $j = 0, 1, ..., q$ $(p, q \geq 0)$. *Then*

$$[\xi + \alpha][\psi + \beta] \in \Lambda \Longleftrightarrow \alpha = \beta = 1.$$

Proof. Let $\mu = c_0c_1...c_n = [\xi + \alpha][\psi + \beta] \in \Lambda$, $n = p + q + 1$, i.e.

$$c_0 = a_0 + \alpha, ..., c_p = a_p + \alpha,\ c_{p+1} = b_{p+1} + \beta, ..., c_n = b_n + \beta.$$

The following equalities hold:

$$x_\mu(k) = x_\xi(k - \alpha) + x_\psi(k - \beta) \ \text{ for } ; k \geq \min\{\alpha, \beta\},$$

$$x_\mu(k) = 0 \ \text{ for } \ k < \min\{\alpha, \beta\},$$

$$s_\mu(r) = s_\xi(r - \alpha) + s_\psi(r - \beta),\ r \geq 0,$$

$$m_\mu = \max\{m_\xi + \alpha, m_\psi + \beta\},$$

(3.11) $$y_\mu(k) = y_\xi(k - \alpha) + y_\psi(k - \beta) \ \text{ for } \ k > \min\{\alpha, \beta\}.$$

(Note that: $x_\lambda(k) = s_\lambda(k) = y_\lambda(k) = 0$ for any negative integer $k(\lambda \in \Lambda)$.)

By (3.5) we have $y_\mu(1) = 1$, $y_\xi(1 - \alpha) \geq 1$, $y_\psi(1 - \beta) \geq 1$ in the case $\alpha, \beta \leq 0$, which is a contradiction to (3.11). Hence, one of the integers α, β must be positive.

Let $\alpha > 0$, $\beta \leq 0$. By (3.11) we get:

$$y_\mu(1) = y_\psi(1 - \beta),\ y_\mu(2) = y_\psi(2 - \beta), ..., y_\mu(\alpha) = y_\psi(\alpha - \beta),$$

$$y_\mu(\alpha + 1) = y_\xi(1) + y_\psi(\alpha + 1 - \beta), ..., y_\mu(m_\mu) = y_\xi(m_\mu - \alpha) + y_\psi(m_\mu - \beta).$$

Having in mind (3.7), we have:

$$n = \sum_{k=0}^{m_\mu} y_\mu(k) = \sum_{k=0}^{m_\xi} y_\xi(k) + \sum_{k=0}^{m_\psi} y_\psi(k) - \sum_{k=0}^{-\beta} y_\psi(k) = p + q - s(-\beta),$$

which implies $s_\psi(-\beta) = -1$. This is contradiction to $s_\psi(-\beta) \geq 0$.

We conclude that $\alpha, \beta > 0$. Now let $0 < \alpha \leq \beta$. Putting $k = \alpha,\ \alpha + 1, ..., m_\mu$ in (3.11), and taking a sum, we get

$$n = \sum_{k=0}^{m_\mu} y_\mu(k) = \sum_{k=0}^{\alpha} y_\mu(k) + \sum_{k=0}^{m_\xi} y_\mu(k) + \sum_{k=0}^{m_\psi} y_\mu(k) = \sum_{k=0}^{\alpha} y_\mu(k) + p + q.$$

Therefore,

$$1 = \sum_{k=0}^{\alpha} y_\mu(k) = y_\mu(1) + ... + y_\mu(\alpha)$$

and, by (3.5), $\alpha = 1$.

Suppose now that $\beta > 1 = \alpha$. Then, by (3.11) we have

$$y_\mu(2) = y_\xi(1) + y_\psi(2-\beta) = y_\xi(1) \leq 1. \tag{3.12}$$

We consider two cases: $\xi \neq 0$ and $\xi = 0$.

At first, take $\xi \neq 0$. Then $x_\mu(1) = x_\xi(0) + x_\xi(1-\beta) = 0$, which means that 1 does not appear in μ. Then $y_\mu(2) = 2$, since, deriving the μ from 0, we have in the first step 11, in the second step either 122 or 221, and in some of the next steps 1 must be changed by 22. This contradiction to (3.12) shows that $\beta = 1$.

Assume now that $\xi = 0$. Then by (3.12) we have $y_\mu(2) = y_0(1) = 0$, i.e. 22 is not derived from 1 in any derivation of μ. So $\mu = 11$, which means $\xi = \psi = 0,\ \alpha = \beta = 1$.

Conversely, let $\alpha = \beta = 1$, and then $\mu = [\xi+1][\psi+1]$. Using some derivations od ξ and ψ from 0, we can get a derivation of μ from 0 too. Namely, assume that the $k-$th and the $k+1-$th steps in some derivations of ξ and ψ are

$$\xi_k = d_0...d_i...d_k,\ \psi_k = e_0...e_j...e_k,$$

$$\xi_{k+1} = d_0\,(d_i+1)(d_i+1)...d_k,\ \psi_{k+1} = e_0...(e_j+1)(e_j+1)...e_k.$$

Assume also that we have a derivation of

$$\mu_{2k+1} = (d_0+1)...(d_i+1)...(d_k+1)(e_0+1)...(e_j+1)...(e_k+1) = [\xi_k+1]][\psi_k+1].$$

Then we can continue:

$$\mu_{2k+2} = (d_0+1)...(d_i+2)(d_i+2)...(d_k+1)(e_0+1)...(e_j+1)...(e_k+1),$$

$$\mu_{2k+3} = (d_0+1)...(d_i+2)(d_i+2)...(d_k+1)(e_0+1)...$$
$$(e_j+2)(e_j+2)...(e_k+1) = [\xi_{k+1}+1][\psi_{k+1}+1]. \ \square$$

4. A combinatorial proof that $\mathcal{FB}$ is a proper $(1,2)-$ subsemigroup of a semigroup

By Theorem 2.1 we have to show that the implications (2.1) and (2.2) hold in $\mathcal{FB} = (FB; f)$. So, let $t, g \in \mathcal{P}(s)$ and $t(u) = g(v)$, where $u = (a, \rho)$, $v = (b, \sigma) \in FB$, $a, b \in B$, and ρ, σ are corresponding sequences of integers. Then for some sequences of integers $\rho_0, ..., \rho_s, \sigma_0, ..., \sigma_s$ we have

$$t(u) = ((a, \rho\rho_0), ..., (a, \rho\rho_s)) = ((b, \sigma\sigma_0), ..., (b, \sigma\sigma_s)) = g(v),$$

which implies $a = b$ and

$$\rho\rho_0 = \sigma\sigma_0,\ \rho\rho_1 = \sigma\sigma_1, ..., \rho\rho_s = \sigma\sigma_s. \tag{4.1}$$

We have still to show that $\rho = \sigma$ and, by (4.1), it is enough to show that ρ and σ are of the same length , i.e. $|\rho| = |\sigma|$. Assume that $|\rho| < |\sigma|$. Then, by (4.1) we have

$$|\rho_0| > |\sigma_0|,\ |\rho_1| > |\sigma_1|, ..., |\rho_s| > |\sigma_s|,$$

which is impossible by Theorem 3.1, since $|\rho_0| ... |\rho_s|$ and $|\sigma_0| ... |\sigma_s|$ are $(1,2)-$sequences from Λ. Thus, (2.1) holds in $\mathcal{FB}$.

Now, let $h \in \mathcal{P}(1, s)$ and $t(u) = h(v, w)$, where u, v are as above and $w = (c, \tau) \in FB$, $c \in B$, τ is a sequence of integers. Then we have

$$t(u) = ((a, \rho\rho_0), ..., (a, \rho\rho_s)) =$$

$$((b, \sigma\sigma_0), ..., (b, \sigma\sigma_n), (c, \tau\tau_0), ..., (c, \tau\tau_m)) = h(v, w),$$

for corresponding sequences of integers $\rho_0, ..., \rho_s$, $\sigma_0, ..., \sigma_n$, $\tau_0, ..., \tau_m$ $(s = n + m + 1)$. Then we have $a = b = c$ and

$$\rho\rho_0 = \sigma\sigma_0, ..., \rho\rho_n = \sigma\sigma_n,\ \rho\rho_{n+1} = \tau\tau_0, ..., \rho\rho_s = \tau\tau_m. \tag{4.2}$$

We note that $\rho' =| \rho_0 | ... | \rho_s |$, $\sigma' =| \sigma_0 | ... | \sigma_n |$, $\tau' =| \tau_0 | ... | \tau_m |$ are $(1,2)-$ sequences from Λ, and by (4.2) we have the equality

$$\rho' = [\sigma' + \alpha][\tau' + \beta],$$

where $\alpha =| \sigma | - | \rho |$, $\beta =| \tau | - | \rho |$. It follows by Theorem 3.2 that $\alpha = \beta = 1$, i.e. $| \sigma |=| \tau |=| \rho | +1$, and by (4.2) we have

$$\sigma = \rho e_1, \ \ \tau = \rho e_2,$$

for some $e_1, e_2 \in \{1,2\}$.

Consider now the $(1, 1+s)-$ operation $t \in \mathcal{P}(s)$. The equality $t(u) = h(v,w)$ implies that there is an operation $t' \in \mathcal{P}(1,s)$ such that

$$t(u) = (t'f)(u)(t'((u,1),(u,2)) = t'((a,\rho 1),(a,\rho 2)).$$

Therefore, the first member of ρ_0 is 1, and the first member of ρ_s is 2, which implies that $e_1 = 1$, $e_2 = 2$. Thus we have $v = (a, \rho 1)$, $w = (a, \rho 2)$, i.e. $v = (u,1)$, $w = (u,2)$. Then $f(u) = (v,w)$, and (2.2) holds in $\mathcal{FB}$ as well.

Remark 5. Here we give a few results and remarks about the $(1,2)-$ sequences. The truth of the next statements is either evident or easily proved.

5.1. Consider a $(1,2)-$sequence $\lambda = a_0 a_1 ... a_{s-1}$ of lenght $s > 1$. Then the following inequalities hold:

$$s[\log_2 s] + 2(s - 2^{[\log_2 s]}) \leq \sum_{i=0}^{s-1} a_i \geq \frac{(s-1)(s+2)}{2},$$

where $[\log_2 s]$ denotes the integer part of $\log_2 s$ (see also [3]).

5.2. We can extend the Theorem 3.2. in one direction. Namely, let $\xi_0, \xi_1, ..., \xi_s$ and $\alpha = a_0 a_1 ... a_s$ are $(1,2)-$ sequences (a_i are integers). Then $[\xi_0 + a_0][\xi_1 + a_1]...[\xi_s + a_s]$ is a $(1,2)-$ sequence as well. The oposite statement is not true: for example, $11, 2222 \in \Lambda$, $212 \notin \Lambda$, but

$$33333333 = [11+2][2222+1][11+2] \in \Lambda.$$

5.3. Define a set Λ_k of $(1,k)-$ sequences ($k \geq 1$) as follows:

$$\lambda_0 = 0, \ \lambda_{11} = 1\underbrace{...}_{k}1, \ \lambda_{21} = 2\underbrace{...}_{k}21\underbrace{...}_{k-1}1, \ \lambda_{22} = 12\underbrace{...}_{k}21\underbrace{...}_{k-2}1, ...,$$

$$\lambda_{2k} = 1\underbrace{...}_{k-1}12\underbrace{...}_{k}2,\ \lambda_{31} = 3\underbrace{...}_{k}32\underbrace{...}_{k-1}21\underbrace{...}_{k-1}1, ...,$$

i.e. a sequence λ_{ij} (of rank i) is derived from a sequence $\lambda_{i-1,m}$ (of rank $i-1$) by using this rule: replace some member j of $\lambda_{i-1,m}$ by the sequence $(j+1)\underbrace{...}_{k}(j+1)$. It is clear that every $(1,k)-$ sequence λ of a rank i has length $|\ \lambda\ | = i(k-1)+1$. Note that the set Λ_1 coincides with the set of nonnegative integers, i.e. the sets Λ_k are generalization of the set of natural numbers.

In the same manner as Theorem 3.1, one can prove this property:

If $a_0a_1...a_{i(k-1)}$ and $b_0b_1...b_{i(k-1)}$ are $(1,k)-$ sequences of rank i, then there exist indices n and m ssuch that $a_n \leq b_n$, $a_m \geq b_m$.

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REZIME

O NIZOVIMA PRIDRUŽENIM SLOBODNIM (1,2)-GRUPOIDIMA

U radu su definisani i izučavani $(1,2)-$nizovi, koji su povezani sa $(1,2)-$operacijama. Korišćenjem tih $(1,2)-$nizova dat je kombinatorni dokaz da je svaki slobodni $(1,2)-$ grupoid prava $(1,2)-$ podsemigrupa neke semigrupe. Dokazano je takodje nekoliko kombinatornih svojstava $(1,2)-$nizova.