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CONGRUENCES ON n-GROUPS

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Abstract

A description of congruences on polyadic groups is given in this paper. Namely, if $\mathbf{Q} = (Q,[\,])$ is an $n+1$-group, and $\mathbf{Q}^\wedge$ is its universal covering group, then each congruence α of $\mathbf{Q}$ can be characterized by an invariant subgroup $\mathbf{H}_\alpha$ of $\mathbf{Q}^\wedge$ such that $\mathbf{H}_\alpha \subseteq \{a_1 \cdot \ldots \cdot a_{n-1} \mid a_v \in Q\}$. Also, if $a \in Q$ and $*$ is an operation on Q defined by $x * y = [xa^{n-2}y]$, then $(Q, *)$ is a group, and each congruence a of $\mathbf{Q}$ is characterized by an invariant subgroup K of $(Q, *)$, such that for each $x \in Q$ $[xa^{n-2}K] = [a^{n-2}Kx]$. It is also shown that it may happen to have an n-group $\mathbf{Q}$ and a congruence α on Q such that neither of the α-equivalence classes is an n-subgroup of $\mathbf{Q}$, and necessary and sufficient conditions are given under which such classes do exist.

In the last section of this paper new and shorter proofs of some propositions of [5] are given using the universal covering group.

0. Let us first state some preliminary definitions and results.

If $[\,] : (x_0, x_1, \ldots, x_n) \mapsto x_0\, x_1 \ldots x_n$ is an associative $n+1$-ary operation on a non empty set Q, then we say that $\mathbf{Q} = (Q;[\,])$ is an $n+1$-semigroup.

The notions of $n+1$-subsemigroup, homomorphism, congruence on an $n+1$-semigroup [1)] are defined in the usual way, so we will not state them explicitly.

If $\mathbf{Q}$ is an $n+1$-semigroup, then the semigroup $\mathbf{Q}^\wedge$, given by the following presentation (in the variety of all semigroups)

$$\mathbf{Q}^\wedge = < Q; \{a = a_0\, a_1 \ldots a_n \mid a = [a_0\, a_1 \ldots a_n]\} > \qquad (0.1)$$

is called the universal covering semigroup of $\mathbf{Q}$. We can assume that $Q \subseteq \mathbf{Q}^\wedge$ and

$$Q^\wedge = Q \cup Q^2 \cup \ldots \cup Q^n$$

where $Q^i = \underbrace{Q\,Q \ldots Q}_{i} = \{a_0, a_1 \ldots a_i \mid a_v \in Q\}$, and that the union is disjoint.

1) In this paper, only for technical reasons, we are using $n+1$-semigroups instead of n-semigroups.

An $n+1$–semigroup $\mathbf{Q} = (Q;[\,])$ is said to be an $n+1$–group iff $(\forall\, a_0, a_1, \dots, a_{n-1}, b \in Q)(\exists x, y \in Q)[a_0 a_1 \dots a_{n-1} x] = b = [y a_0 a_1 \dots a_{n-1}]$.

There are several axiom sistems of $n+1$–groups (see, for example, [1]), and we note that an $n+1$–semigroups is an $n+1$–group iff its universal covering semigroup is a group ([1]).

A homomorphism φ from an $n+1$–semigroup $\mathbf{Q}$ into an $n+1$–semigroup $\mathbf{Q}'$ can be uniquely extended to a homomorphism $\varphi^\wedge : Q^\wedge \to Q'^\wedge$, such that φ is the restriction of $\varphi^\wedge$ on $\mathbf{Q}$. Namely, $\varphi^\wedge$ is defined in the following way:

$$a = a_1 \dots a_i \in Q^i, \quad a_v \in Q \Rightarrow \varphi^\wedge(a) = \varphi(a_1) \dots \varphi(a_i)\,.$$

We note that if φ is surjective or bijective, then $\varphi^\wedge$ has the corresponding property, but it can happen φ to be injective mapping and $\varphi^\wedge$ not to be such one ([4]). Thus, in general, if P is an $n+1$–subsemigroup of an $n+1$–semigroup $\mathbf{Q}$, $P^\wedge$ may not be an $n+1$–subsemigroup of $\mathbf{Q}^\wedge$. But if we consider $n+1$–groups the following statement holds:

if H is an $n+1$–subgroup of the $n+1$–group $\mathbf{Q}$, then $H^\wedge$ is a subgroup of $\mathbf{Q}^\wedge$ ([4]).

1. Let $\mathbf{Q} = (Q;[\,])$ be an $n+1$–semigroup, and $\mathbf{Q}^\wedge$ its universal covering semigroup. Let α be a congruence on $\mathbf{Q}$. Then $\varphi = \operatorname{nat}\alpha$ is an epimorphism from $\mathbf{Q}$ into $\mathbf{Q}_{/\alpha}$, and $\varphi^\wedge = (\operatorname{nat}\alpha)^\wedge$ is an epimorphism from $\mathbf{Q}^\wedge$ into $(\mathbf{Q}_{/\alpha})^\wedge$. If $\lambda^\wedge = \ker(\operatorname{nat}\alpha)^\wedge$, then $\alpha^\wedge$ is a congruence on the semigroup $\mathbf{Q}^\wedge$ with the property

$$x_1 \dots x_i\, \alpha^\wedge\, y_1 \dots y_j \Rightarrow i = j \,(\operatorname{mod} n), \quad (x_v, y_\lambda \in Q)$$ [2)]

1.1°. Let $\mathbf{Q} =$ be an $n+1$–semigroup, and $\mathbf{Q}^\wedge$ its universal coverting semigroup. If β is a congruence on $\mathbf{Q}^\wedge$, then $\alpha = \beta|_Q$ is a congruence on $\mathbf{Q}$, such that $\alpha^\wedge \subseteq \beta$. ◊

1.2°. Let $\mathbf{Q}$ be an $n+1$–semigroup, and $\mathbf{Q}^\wedge$ its universal covering semigroup. Then β is a congruence on $\mathbf{Q}^\wedge$ with the property

$$\alpha = \beta|_Q \;\&\; x_1 \dots x_i \beta y_1 \dots y_j \Rightarrow i = j \text{ iff } a^\wedge = \beta\,. \qquad \Diamond$$

2. Let us now give a description of the subgroups of the universal covering group $\mathbf{Q}^\wedge$ of an $n+1$–group $\mathbf{Q}$.

2.1°. H is a subgroup of $\mathbf{Q}^\wedge$ iff there exists a natural number r, $1 \le r \le n$, such that $r|n$, and $H = H_r \cup H_r^2 \cup \dots \cup H_r^q$, where $H_r = H \cap Q^r$, $n = rq$. In this case H_r is a $q+1$–subgroup of H and $H_r^\wedge = H$([3]). ◊

2.2°. H is an invariant subgroup of $\mathbf{Q}^\wedge$ iff for each x in Q, $xH_r = H_r x$([3]). ◊

Let us note another property of $\mathbf{Q}^\wedge$. Namely, if $a \in Q$, then $Q^\wedge = Q \cup aQ \cup \dots \cup a^{n-1}Q$, i.e. $Q^i = a^{i-1}Q$. Then $(a^i x)(a^j y) = a^{i+j+1} z$, where z is the solution of the equation $[a^n z] = [a^{n-j-1}\, x a^j y]$, and $i+j+1$ is counted by modulo n. Specially, $(a^{n-1}x)(a^{n-1}y) = a^{n-1}z$, where $z = [x a^{n-1} y]$.

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2) Here, and further on, we will assume that $x_v, y_v \in Q$ and $i, j \le n$.

3. Let us now state some properties of congruences of $n+1$–groups. Note that, further on with $\mathbf{Q}$ will be denoted an $n+1$–group, and with $\mathbf{Q}^\wedge$ its universal covering group. We will often write „subgroup" instead of „„$n+1$–subgroup" whenever the meaning will be clear from the context.

3.1°. Let α be a congruence on $\mathbf{Q}$. Then $(Q_{/\alpha})^\wedge \cong Q^\wedge_{/\alpha^\wedge}$, and

$$x_1 \dots x_i \alpha^\wedge y_1 \dots y_j \Leftrightarrow$$
$$\Leftrightarrow i=j \& (\exists c_0, \dots, c_{n-i} \in Q)[c_0 \dots c_{n-i} x_1 \dots x_i] \alpha [c_0 \dots c_{n-i} y_1 \dots y_i]. \quad (*)$$

Conversely, if β is a congruence on $\mathbf{Q}^\wedge$ and $\alpha = \beta|_{\mathbf{Q}}$, then $\alpha^\wedge \subseteq \beta$, and $\alpha^\wedge = \beta \Leftrightarrow a_1 \dots a_i \, \beta \, b_1 \dots b_j \Rightarrow i = j$. ◇

The fact that $\mathbf{Q}^\wedge$ is a group allows us to describe $\alpha^\wedge$ by a corresponding normal subgroup H of $\mathbf{Q}^\wedge$. Namely:

3.2°. Let α be congruence on $\mathbf{Q}$, $H^\alpha = \{x \in Q^\wedge \mid x \alpha^\wedge e\}$. [3] Then $H^\alpha \subseteq Q^n$ is an invariant subgroup of $\mathbf{Q}^\wedge$, and $Q_{/\alpha} = \{xH^\alpha \mid x \in Q\}$.

Conversely, if K is an invariant subgroup of $\mathbf{Q}^\wedge$ such that $K \subseteq Q^n$, then the corresponding congruence β^K on $\mathbf{Q}^\wedge$ induces a congruence $\alpha = \beta^K|_{\mathbf{Q}}$ on Q, such that $\alpha^\wedge = \beta^K$ ◇

A more general statement of this property is the following: if L is an invariant subgroup of $\mathbf{Q}^\wedge$, then $K = L \cap Q$ is an invariant subgroup of $\mathbf{Q}^\wedge$, such that $B^L|_{\mathbf{Q}} = \beta^K|_{\mathbf{Q}}$, and in this case $(\beta^K|_{\mathbf{Q}})^\wedge \subset \beta^L$ whenever L is not a subset of Q^n. [4]

4. Further on we will give a description of the congruences of an $n+1$–group $\mathbf{Q}$ by corresponding non empty subsets of Q.

4.1°. Let $\mathbf{Q}$ be an $n+1$–group, $a \in Q$ and let us define a binary operation $*$ in Q by:

$$x * y = [x a^{n-1} y].$$

Then $(Q;*)$ is a group isomorphic to $\mathbf{Q}^n$ by the isomorphism φ: $x \mapsto a^{n-1}x$. ◇

It is easy to prove that each congruence α on $(Q;[\,])$ is a congruence on $(Q;*)$ as well. Thus it induces an invariant subgroup $K = K^{a,\alpha}$ of $(Q;*)$. Consequently,

$$Q_{/\alpha} = Q_{/K} = \{x * K \mid x \in Q\} = \{[x a^{n-1} K] \mid x \in Q\}$$

and

$$[x a^{n-1} K] = [K a^{n-1} x].$$

Thus, we can describe the α–classes by special subsets of Q, which depend on the choice of the element a.

Conversely, Let K be an invariant subgroup of $(Q;*)$. Then K induces an equivalence relation α on Q defined by $Q_{/\alpha} = Q_{/K}$. Now, the question

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3) e is the neutral element of the group $\mathbf{Q}^\wedge$.

4) This is another way of stating Th. 2.7 of [5].

arises to find conditions such that α is a congruence on $(Q;[\,])$. We will answer this question considering the group Q^n, isomorphic to $(Q;*)$.

4.2°. α is a congruence on $\mathbf{Q}$ iff $a^{n-1}K$ is an invariant subgroup of $\mathbf{Q}^n$, i.e. iff
$$(\forall x \in Q) x a^{n-1} K = a^{n-1} K x . \qquad \diamondsuit$$

5. Let us recall that if H is a subgroup of an $n+1$–group $\mathbf{Q}$ with the property
$$(\forall x \in Q)[xH^n] = [H^n x],$$
then we say that H is semiinvariant subgroup of $\mathbf{Q}$ ([2]). In this case H is a class of a congruence α on $\mathbf{Q}$ defined by
$$Q_{/\alpha} = Q_{/H} = \{[xH^n] \mid x \in Q\}.$$

The following property is also true:

The subgroup H of $\mathbf{Q}$ is semiinvariant in $\mathbf{Q}$ iff H^n is an invariant subgroup of $\mathbf{Q}^\wedge$ ([2]).

It is natural to ask the question whether every congruence on $\mathbf{Q}$ is obtained by a semiinvariant subgroup of $\mathbf{Q}$. The answer of this question is negative. Let us give some examples to illustrate this fact.

Example **1**. Let $Q = (2k+1)Z$ be the set of odd integers. We define a ternary operation by $[xyz] = x + y + z$, and obtain a 3–group, such that $Q^\wedge \cong (Z, +)$. Let α be the congruence on Q such that
$$Q_{/\alpha} = \big\{\{4k+1 \mid k \in Z\},\ \{4k+3 \mid k \in Z\}\big\}.$$
In this case there is no α–class that is a subgroup of $\mathbf{Q}$.

Example **2**. Let $\mathbf{Q}$ be the same as in Ex. **1**, and α be the congruence on $\mathbf{Q}$ such that
$$Q_{/\alpha} = \big\{\{6k+1 \mid k \in Z\},\ \{6k+3 \mid k \in Z\},\ \{6k+5 \mid k \in Z\}\big\}.$$
Then $\{6k+3 \mid k \in Z\}$ is the unique α–class that is a subgroup of $\mathbf{Q}$.

Example **3**. Let $Q = \{a_1, a_3, b_1, b_3\} \subseteq D_4$, and let define an operation $[\,]$ by $[xyz] = x \circ y \circ z$, where „$\circ$" is the operation in the dihedral group D_4. Then $\mathbf{Q}$ is a 3–group, and $\mathbf{Q}^\wedge = D_4$. If we define α to be the congruence on $\mathbf{Q}$ such that
$$Q_{/\alpha} = \big\{\{a_1, a_3\},\ \{b_1, b_3\}\big\},$$
then both the α–classes are subgroups of $\mathbf{Q}$.

These examples initiate a problem of finding conditions under which an α–class is a subgroup of $\mathbf{Q}$. The answer is given by the following property:

5.1°. Let $\mathbf{Q}$ be an $n+1$–group, let α be a congruence on $\mathbf{Q}$, $a \in Q$, and $H = H^{a,\alpha}$ be the non empty subset of Q induced by α and a (as in in **4**.). Then: $[xa^{n-1}H]$ is a subgroup of $\mathbf{Q}$ iff $[x^{n+1}] \in [xa^{n+1}H]$, and in this case $[xa^{n-1}H]$ is a semiinvariant subgroup of $\mathbf{Q}$. $\diamondsuit$

6. In this last section of this paper we will give new proofs of some of the results of [5], using the universal covering group of an n–group $\mathbf{Q}$.

Let α be a congruence on an $n+1$–group $\mathbf{Q}$. Then the α–equivalence class containing the element $z \in Q$ is called a z–ideal. Using the results in the previous section it is easily seen that a z–ideal I of an $n+1$–group $\mathbf{Q}$ could be expressed in the form $I = zB$, where B is an invariant subgroup of

$\mathbf{Q}^\wedge$, such that $B \subseteq Q^n$. Using this fact and the properties of the universal covering group $\mathbf{Q}^\wedge$ we can easily prove the following:

6.1°. (Th. **2.5** in [5].) If α and β are two congruences of an $n+1$–group $\mathbf{Q}$, then

$$(\exists z \in Q) z^\alpha = z^\beta \Rightarrow \alpha = \beta\,. \qquad \diamondsuit$$

If I is a z–ideal, clearly $z \in I$ and $I = zB$ for some normal subgroup B of $\mathbf{Q}^\wedge$ which is a subset of Q^n. Then, if $x, y \in I$, $x = zb_1$, $y = zb_2$, we have $[zb_1 z^{1-n} z^{n-2} zb_2] = [zb_1 b_2] \in zB = I$. If $x, y \in Q$ and $[x\,\overline{y}\,yy^{n-2}z \in zB$, where $\overline{y}$ is y^{1-n}, using the fact that $B \le Q^\wedge$ we obtain that $[x\,\overline{x}\,x^{n-2}z] \in I = zB$ as well. Using the fact that $B \lhd Q^\wedge$ we easly check that if $y, x_1, \dots, x_n \in Q$, $[x_1^n z] \in zB$, then $[[yx_1^n]y^{-1}z] \in zB$, i.e. $[[yx_1^n]\,\overline{y}\,y^{n-2}z] \in zB$. Thus we have proven the „if" part of the following:

6.2°. (Th. **2.6** of [5]). Let $\mathbf{Q}$ be an $n+1$–group and $z \in Q$. The following two conditions are equivalent:

(i) I is z–ideal;

(ii) a) $z \in I$;

b) $x, y \in I \Rightarrow [x\,\overline{z}\,z^{n-2}y] \in I$;

c) $x, y \in Q$ and $[x\,\overline{y}\,y^{n-2}z] \in I \Rightarrow [y\,\overline{x}\,x^{n-2}z] \in I$;

d) $y, x_1, \dots, x_n \in Q$, $[x_1^n] \in I \Rightarrow [[yx_1^n]\,\overline{y}\,y^{n-2}z] \in I$.

To prove the converse we first show that if a)–d) hold, and if $B = z^{-1}I$, i.e. $I = zB$, then $B \subseteq Q^n$. Uing a)–d) then we prove that $B \lhd Q^\wedge$. Namely, a) implies that the unity of $\mathbf{Q}^\wedge$ is in B, b) and c) imply that B is a subgroup, and d) that B is an invariant subgroup of $\mathbf{Q}^\wedge$. $\diamondsuit$

These two theorems give another characterization of a congruence of an $n+1$–group $\mathbf{Q}$, i.e. each congruence α of an $n+1$–group $\mathbf{Q}$ is uniquely determined by its z–ideal.

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КОНГРУЕНЦИИ НА n-ГРУПИ

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Резиме

Во овој труд е даден опис на конгруенции на полиадични групи. Имено, ако $\mathbf{Q} = (Q;[\,])$ е n–група, а $\mathbf{Q}^\wedge$ е нејзината универзална покривачка група, тогаш секоја конгруенција α на $\mathbf{Q}$ е окарактеризирана со нормална подгрупа $H_\alpha \subseteq \{a_1 \dots a_{n-1} \mid a_v \in Q\}$. Исто така, ако $a \in Q$ и $*$ е операција на Q дефинирана со $x * y = [xa^{n-2}\, y]$, тогаш $(Q;*)$ е група и секоја конгруенција α на Q е окарактеризирана со нормална подгрупа K од $(Q;*)$ таква што за секој $x \in Q$, $[xa^{n-2}\, K] = [a^{n-2}\, Kx]$. Покажано е и дека може да се најде n–група $\mathbf{Q}$ и конгруенција α на $\mathbf{Q}$ такви што ниедна класа на еквивалентноста α да не биде n–подгрупа од $\mathbf{Q}$, и најдени се нужни и доволни услови при кои класа на еквивалентноста α е n–подгрупа од $\mathbf{Q}$.

На крајот од трудот дадени се нови, пократки докази на некои својства од [5] преку универзалната покривачка група.

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