$$k+[u]\geq 0,\ k+m+[u]\geq 0\Rightarrow \left(u^{(k)}\right)^{(m)}=u^{(k+m)} \tag{1.4}$$

$$k+[u]\geq 0,\ m-k+[v]\geq 0\Rightarrow (u^{(k)}=v^{(m)}\Leftrightarrow u=v^{(m-k)} \tag{1.5}$$

$$k+[u]\geq 0\Rightarrow \left(u^{(k)}\in R\Leftrightarrow u\in R\right) \tag{1.6}$$

We will also use the following two lemmas, which can be also easily shown.

Lemma 1.1. If $\varphi\colon \mathbf{Q}\to\mathbf{G}$ is a homomorphism, then

$$x\in Q,\quad m\geq 0\Rightarrow \varphi\left(x^{(m)}\right)=\varphi(x)^{(m)}.\ \square$$

Lemma 1.2. If $\mathbf{G}\in\mathcal{V}$ and $x,\ y\in G$, $m\geq 0$, then

$$(xy)^{(m)}=x^{(m)}y^{(m)}.\ \square$$

Now we can prove *Theorem* 1.

First, if $u\in R$ is such that $[u]=m$, then by (1.4) we have

$$u*u=\left(u^{(-m)}u^{(-m)}\right)^{(m)}=\left(\left(u^{(-m)}\right)^{(1)}\right)^{(m)}=u^2. \tag{1.7}$$

Let $u,\ v\in R$ be such that $u\neq v$, and $\min\{[u],\ [v]\}=m$. Then $\left[u^{(-m)}\right]=0$ or $\left[v^{(-m)}\right]=0$, wich implies $u^{(-m)}v^{(-m)}\in R$, and by (1.6) we obtain $u*v\in R$. Thus:

$$u,\ v\in R\Rightarrow u*v\in R, \tag{1.8}$$

i.e. $\mathbf{R}=(R,\ *)$ is a groupoid.

Moreover, if $\min\{[u],\ [v]\}=m$, then:

$$\begin{aligned}(u*v)*(u*v)&=(u*v)^2=(u*v)^{(1)}=\left(u^{(-m)}v^{(-m)}\right)^{(m+1)},\\ (u*u)*(v*v)&=u^2*v^2=\left((u^2)^{(-m-1)}(v^2)^{(-m-1)}\right)^{(m+1)}\\ &=\left(u^{(-m)}v^{(-m)}\right)^{(m+1)},\end{aligned}$$

and this implies that $\mathbf{R}\in\mathcal{V}$.

If $u,\ v\in R$ are such that $uv\in R$, then $u=v$ or $\min\{[u],\ [v]\}=0$, and thus we have:

$$u,\ v,\ uv\in R\Rightarrow u*v=uv, \tag{1.9}$$

and so B is a generating set of R. Clearly, B is the set of primes in $\mathbf{R}$.