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NOTES ON HOMOMORPHISMS AND FREE OBJECTS IN A CLASS OF GENERALIZED ALGEBRAS

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A b s t r a c t: The notion of φ-algebra, which is a generalization of the usual notion of algebra, is introduced in [3]. As in [2], two kinds of substructures are defined, usual and strong ones, and one kind of homomorphism. As a consequence we obtain two kinds of free objects (free and weakly free φ-algebras) in the class of φ-algebras of a given type. A complete description of the free objects of both kinds is given in [3]. Homomorphisms defined in [3] have some "bad" properties. In fact, they do not keep the structure of the given φ-algebras in a way that we expect from homomorphisms (of course, we do not expect from homomorphisms to remember everything, but, we expect this, at least, from isomorphisms and that was not the case in [3]). Here is given a new definition of homomorphisms and these new homomorphisms cooperate much better with the given structures. This new defined class of homomorphisms is a subclass of the old one, so some of the results are just transfered from [3]. Let us mention that the description of the free objects remains the same as in [3].

0. Introduction

We will quote several defenitions and propositions as they appear in [3]:

Def. 0.1. Let A be a nonempty set, **B** an algebra of type Ω[1] and φ: A→B a surjective mapping. The ordered triple $\boldsymbol{A}$ = (A, φ, B) is called φ-algebra of type Ω. The set A is called carrier of the φ-algebra A.

It should be mentioned that the name φ-algebra does not depend on the name of defined surjective mapping. For example, if C is a set, **D** is an algebra of type Ω and ψ: C→D is a surjective mapping, then the ordered triple *C*=(C, ψ, **D**) is also called φ-algebra of type Ω .

Further, φ-algebras (**A**, φ, **B**) and (**C**, ψ, **D**) of type Ω will be denoted by ***A*** and *C*, respectively.

If **B** is an algebra of type Ω, then **B** can be considered as φ-algebra, were A = B and $\varphi = 1_B$. Whence, the notion of φ-algebra is a generalization of the notion of algebra.

[1] $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \ldots$, where Ω_1 consists of symbols of arity i.

Def. 0.2. A subset $S \subseteq A$ is a φ-subalgebra of $\boldsymbol{A}$ if φS is a subalgebra of B[2].

S is strong φ-subalgebra of $\boldsymbol{A}$ if there is some subalgebra T of **B** such that $S = \varphi^{-1}T$.

If the mapping φ is bijective, there is no distinction between φ-subalgebras and strong φ-subalgebras. Especially, if A = B and $\varphi = 1_B$, then the subalgebras of **B**, and only they, are φ-subalgebras and strong φ-subalgebras of $\boldsymbol{A}$.

From now on, we will call φ-subalgebras and strong φ-subalgebras just subalgebras and strong subalgebras, keeping in mind that when we talk about subalgebra or strong subalgebra of φ-algebra we actually talk about φ-subalgebra or strong φ-subalgebra. It is easy to see that every strong subalgebra is a subalgebra and there are examples which show that the converse does not hold. Also, the intersection of strong subalgebras is a strong subalgebra, the union of a chain of strong subalgebras is a strong subalgebra, the union of a chain of subalgebras is a subalgebra and there are examples of subalgebras whose intersection is not a subalgebra.

Def. 0.4. Let X be a subset of A. We say that X is a generating (weakly generating) subset of $\boldsymbol{A}$ if A is the only subalgerba (strong subalgebra) that contains X.

If φ is a bijektive mapping, there is no distinction between generating and weakly generating subsets. Especially, if A = B and $\varphi = 1$, then the generating subsets of **B** (and only they) are the generating and weakly generating subsets of $\boldsymbol{A}$. Every generating subset is a weakly generating subset and there are examples of weakly generating subsets that are not generating subsets.

Def. 0.5. Let $\boldsymbol{A} = (A, \varphi, \mathbf{B})$ and $\boldsymbol{C} = (C, \psi, \mathbf{D})$ be two φ-algebras of the same type Ω. A mapping $\alpha\colon A \to C$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$ if for every $n \in \mathbf{N}$, n-ary function symbol $\omega \in \Omega_n$, $a_1, \dots, a_n$, $a \in A$ the following implication holds:

$$\omega^B(\varphi a_1, \dots, \varphi a_n) = \varphi a \Rightarrow \omega^D(\psi\alpha a_1, \dots, \psi\alpha a_n) = \psi\alpha a \, ,$$

$\alpha\colon A \to C$ is isomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$ if α is a bijective mapping such that α and α^{-1} are homomorphisms.

If $\boldsymbol{A}$ and $\boldsymbol{C}$ are algebras, then homomorphisms defined in this way are the usual ones. Also, it is obvious that φ is a surjective homomorphism from $\boldsymbol{A}$ to the φ-algebra $(B, 1_B, \mathbf{B})$. If **D** is trivial algebra, then every mapping $\alpha\colon A \to C$ is a homomorphism. The identity mapping $1\colon A \to A$ is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}$. If α is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$, then α^{-1} is an isomorphism from $\boldsymbol{C}$ to $\boldsymbol{A}$. The composition of homomorphisms (isomorphisms) is a homomorphism (isomorphism). Homomorphic images of subalgebras are subalgebras. Inverse homomorphic images of strong subalgebras are strong subalgebras. There are examples in [3] which show that the inverse homomorphic image of a subalgebra is not always a subalgebra 7and tlte homomorphic image of a strong subalgebra is not always a strong subalgebra.

[2] Nonempty subset closed for operations on B.

1774

1. A note on homomorphisms

There is an example in [3] of isomorphism α: $\boldsymbol{A}\to\boldsymbol{C}$ such that corresponding algebras **B** and **D** are not isomorphic (they have different number of elements) and there are two elements a and b in A that have same image under φ, but their images under the isomorphism α have not same image under ψ. We have concluded there that homomorphisms (and isomorphisms as well) are not compatible with the given mappings φ and ψ.

So, we will give here a new definition of homomorphisms that better "preserves" the structure of φ-algebras.

Def. 1.1. Let $\boldsymbol{A}$ = (A, φ, **B**) and $\boldsymbol{C}$ = (C, ψ, **D**) be two φ-algebras of the same type Ω. A mapping α: A$\to$C is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$ if there is a homomorphism β: **B**$\to$**D** such that $\beta\varphi = \psi\alpha$, i.e., the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \alpha\downarrow & & \downarrow\beta \\ C & \xrightarrow[\psi]{} & D \end{array}$$

Now, it is obvious that if two elements from A have same image under φ then for every homomorphism α their images under α have the same images under ψ.

The homomorphism β that corresponds to the homomorphism α in the given definition is unique because φ is a surjective mapping. But, the same homomorphism β may correspond to several homomorphisms α. For example, if **D** is trivial algebra, then every mapping α: A$\to$C is a homomorphism and the trivial homomorphism β from **B** to **D** corresponds to all of them. If, in addition, **B** is not trivial and A and C have the same cardinal number, then every bijective mapping from A to C is a bijective homomorphism, but their inverses are not homomorphisms. We had the same situation before (in [3]), and we have defined isomorphisms in a categorical way, so we will do it here again:

Def. 1.2. A mapping α: A$\to$C is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$ if α is bijective mapping such that α and α^{-1} are homomorphisms.

If $\boldsymbol{A}$ and $\boldsymbol{C}$ are algebras, then homomorphisms (isomorphisms) defined in this way are the usual ones. Also, it is obvious that φ is a surjective homomorphism from $\boldsymbol{A}$ to the φ-algebra (B, 1_B, **B**). The identity mapping 1_A: A$\to$A is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}$ and the only corresponding homomorphism β: **B**$\to$**B** is the identity mapping 1_B. If α is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$ then α^{-1} is an isomorphism from $\boldsymbol{C}$ to $\boldsymbol{A}$. If α is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$ with corresponding homomorphism β: $\boldsymbol{B}\to$**D**, $\boldsymbol{G}$ = (G, ζ, **H**) is another φ-algebra, γ: $\boldsymbol{C}\to\boldsymbol{G}$ is a homomorphism with corresponding homomorphism δ: **D**$\to$**H**, then the composition $\gamma\alpha$: A$\to$G is a homomorphism with corresponding homomorphism $\delta\beta$: **B**$\to$**H**:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \alpha\downarrow & & \downarrow\beta \\ C & \xrightarrow[\psi]{} & D \\ \gamma\downarrow & & \downarrow\delta \\ G & \xrightarrow[\zeta]{} & H \end{array}$$

So, the composition of homomorphisms is a homomorphism. Of course, the composition of isomorphisms is also a isomorphism (if α and γ are isomorphisms then α, γ, α^{-1} and γ^{-1} are homomorphisms, and so are $\gamma\alpha$ and $(\gamma\alpha)^{-1} = \alpha^{-1}\gamma^{-1}$). Let us consider the isomorphism case more carefully. Let α: $A \to C$ be an isomorphism from A to C with corresponding homomorphism β: **B**$\to$**D**, and let the homomorphism β': **D**$\to$**B** correspond to the homomorphism α^{-1}: $C \to A$. The composition $\alpha\alpha^{-1}$: $C \to C$ is the identity isomorphism 1_C, so the corresponding composition $\beta\beta'$: **D**$\to$**D** must be the identity isomorphism 1_D. Also, the composition $\alpha^{-1}\alpha$: $A \to A$ is the identity isomorphism 1_A, so the corresponding composition $\beta'\beta$: **B**$\to$**B** must be the identity isomorphism 1_B. So, β and β' are mutually inverse isomorphisms. Thus, we have the following:

Proposition 1.3. If α: $A \to C$ is an isomorphism from A to C, then the corresponding homomorphism β: **B**$\to$**D** is an isomorphism and its inverse β^{-1}: **D**$\to$**B** corresponds to the inverse isomorphism α^{-1}: $A \to C$. ■

We can see that isomorphisms keep the structure of φ-algebras in a standard way, i.e., isomorphisms are just renaming of the elements. This was not the case with isomorphisms defined as in [3] (there were isomorphic φ-algebras A and C with nonisomorphic algebras **B** and **D**).

Finaly, we will show that these new homomorphisms satisfy the condition for homomorphism given in [3], i.e., they are homomorphisms by the old definition. Thus, we will be able to use some results from [3] without proofs.

Proposition 1.4. Let α: A$\to$C is a homomorphism. Then, for every $n \in N$, n-ary function symbol $\omega \in \Omega_n$, $a_1,\dots, a_n$, $a \in A$ the following implication holds:

$$\omega^B(\varphi a_1,\dots,\varphi a_n) = \varphi a \Rightarrow \omega^D(\psi\alpha a_1,\dots,\psi\alpha a_n) = \psi\alpha a.$$

Proof. Let $\omega^B(\varphi a_1,\dots,\varphi a_n) = \varphi a$ and β: B$\to$D be the corresponding homomorphism from **B** to **D**. Then

$$\omega^D(\psi\alpha a_1,\dots,\psi\alpha a_n) = \omega^D(\beta\varphi a_1,\dots,\beta\varphi a_n) = \beta\omega^B(\varphi\alpha_1,\dots,\varphi\alpha_n) = \beta\varphi a = \psi\alpha a \quad ■$$

It is clear that our new class of homomorphisms is smaller than the old one. For example, there were homomorphisms α: $A \to C$ in [3] that do not satisfy the condition

$\beta\varphi = \psi\alpha$ for any mapping β: B→D. In fact, this was the motivation for our new definition of homomorphisms.

It was proven in [3] that homomorphic images of subalgebras are subalgebras and inverse homomorphic images of strong subalgebras are strong subalgebras. According to proposition 1.4. the same is true even with the new definition of homomorphisms. The same examples as in [3] still show that the inverse homomorphic image of a subalgebra is not always a subalgebra and the homomorphic image of a strong subalgebra is not always a strong subalgebra.

2. A note on free objects

There is almost no need for this part of the paper because all results are the same as in [3], i.e., our new definition of the notion of homomorphism does not affect the results in [3] about the existence of free objects and their description. The proofs are completely the same or sometimes shorter. This is because in some proofs in [3] we first show that there are a mapping α: A→C and a homomorphism β: **B→D** such that $\beta\varphi = \psi\alpha$ and then show that the mapping α is a homomorphism. Now, in such cases α is a homomorphism by definition. Anyway, we will repeat the main results of the corresponding section of [3] in the shortest possible way.

Def. 2.1. ***A*** is a free (weakly free) φ-algebra with basis (weak basis) $A' \subseteq A$ if A' is a generating (weakly, generating) subset of ***A*** and for every φ-algebra ***C*** of the same type and every mapping α': A'→C, α' can be extended to a homomorphism α from ***A*** to ***C***.

Since every generating subset of A is weakly generating subset of A, it is clear that if ***A*** is a free φ-algebra with basis A', then ***A*** is a weakly free φ-algebra with weak basis A'.

Def. 2.2. A φ-algebra ***A*** is called pumped free algebra with basis $A' \subseteq A$ if **B** is free algebra with basis $B' = \varphi A'$ and $\varphi^{-1}\varphi\alpha = a$[3], for every $a \in A'$.

Finally, we quote two theorems from [3] that completely describe the free objects of both kinds in the class of all φ-algebras of a given type in the case when the type Ω is not trivial.

Theorem 2.3. (4.8. in [3]). Let Ω be a nontrivial type. A φ-algebra ***A*** of type Ω is a weakly free φ-algebra with weak basis A' if and only if it is a pumped free algebra with basis A'.

[3] The condition $\varphi^{-1}\varphi a = a$ should be written as $\varphi^{-1}\varphi(\{a\}) = \{a\}$, but we will keep the first notation.

Theorem 2.4. (4.9. in [3]). Let Ω be a nontrivial type. A φ-algebra $\boldsymbol{A}$ of type Ω is a free φ-algebra with basis A' if and only if **B** is a free algebra with basis B' = φA' and φ is a bijective mapping.

Theorems 2.3. and 2.4. hold even in the case when the type is trivial. This is the only difference in the description of the free objects here and in [3].

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Резиме

ЗАБЕЛЕШКИ ЗА ХОМОМОРФИЗМИТЕ И СЛОБОДНИТЕ ОБЈЕКТИ ВО ЕДНА КЛАСА ОБОПШТЕНИ АЛГЕБРИ

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Поимот на φ-алгебра, кој е обопштување на вообичаениот поим на алгебра, е воведен во [3]. Како и во [2], дефинирани се два вида подструктури, обични и јаки, и еден вид хомоморфизми. Како последица добиваме два вида слободни објекти (слободни и слабо слободни φ-алгебри) во класата на φ-алгебри од даден тип. Комплетен опис на слободните објекти од двата вида е даден во [3]. Хомоморфизмите дефинирани во [3] имаат некои „лоши" особини. Всушност, тие не ја чуваат структурата на дадените φ-алгебри на начин кој го очекуваме од хомоморфизмите (се разбира, не очекуваме од хомоморфизмите да помнат сè, но, тоа го очекуваме, барем, од изоморфизмите, а тоа не беше случај во [3]). Тука е дадена нова дефиниција на хомоморфизам и овие нови хомоморфизми соработуваат многу подобро со дадените структури. Новата класа хомоморфизми е подкласа на старата, па некои од резултатите се само пренесени од [3]. Да напоменеме дека описот на слободните објекти останува ист како и во [3].