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ON THE REDUCTION OF THE NUMBER OF VARIABLES IN IDENTITIES

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ABSTRACT. Here we present an algorithm for reduction of the number of variables in an identity. Let :

(1) $$t_1(x_{i_1},\dots,x_{i_n}) = t_2(x_{j_1},\dots,x_{j_m})$$

be an identity, where t_1, t_2 are terms in a signature Ω, $x_1, x_2, \dots$ are variables, n, m are integers such that $n \geq m \geq 0$ and $n > 1$.

Then (1) is equivalent to a reduced finite system of identities. *Reduced system* is defined as the one which has a minimal number of variables, as defined in the text.

Let Ω be a functional signature, i.e. Ω consists of functional symbols of various arities. Denote by $X = \{x_1, x_2, \dots\}$ the set of variables, and for any Ω-term t let $t(x_{i_1},\dots,x_{i_n})$ denote that t is a term built up from the variables $x_{i_1},\dots,x_{i_n}$ (not necessarily distinct but all occurring in t). By an Ω-identity we mean an equality of the form (1), where t_1, t_2 are Ω-terms. Further on we will omit the prefix Ω and instead of (1) we will simply write $t_1 = t_2$.

We say that the identity (1) is a consequence of the identity $t_3(x_{p_1},\dots,x_{p_k}) = t_4(x_{q_1},\dots,x_{q_s})$ if and only if for any Ω-algebra $\mathcal{A} = (A, \Omega)$ we have

$$t_3^{\mathcal{A}}(a_{p_1},\dots,a_{p_k}) = t_4^{\mathcal{A}}(a_{q_1},\dots,a_{q_s}) \text{ for any } a_\lambda \in A$$

$$\Rightarrow t_1^{\mathcal{A}}(a_{i_1},\dots,a_{i_n}) = t_2^{\mathcal{A}}(a_{j_1},\dots,a_{j_m}) \text{ for any } a_\lambda \in A$$

where by $t^{\mathcal{A}}$ we denote the operation on $\mathcal{A}$ derived from the term t. Then we write $t_3 = t_4 \models t_1 = t_2$.

If $t_3 = t_4 \models t_1 = t_2$ and $t_1 = t_2 \models t_3 = t_4$, then we say that the identities $t_1 = t_2$ and $t_3 = t_4$ are equivalent. More generally, if Σ and Λ are sets of identities, then $\Sigma \models \Lambda$ asserts that any algebra satisfying all identities in Σ satisfies all identities in Λ too. If $\Sigma \models \Lambda$ and $\Lambda \models \Sigma$, then we say that Σ and Λ are equivalent.

We shall use the following notation :

$$var(t(x_{i_1},\dots,x_{i_n})) = \{x_{i_1},\dots,x_{i_n}\}, \quad var(t_1 = t_2) = var(t_1) \cup var(t_2).$$

By $t[x \to y]$ we denote the term obtained from t replacing all appearences of x in t by y $(x, y \in X)$. By $\equiv$ we denote the formal equality of terms. For example, $t[x \to y] \equiv t$ if $x \notin var(t)$.

The following proposition is straightforward.

Proposition 1. *For all $x, y \in X$ we have*

$$t_1 = t_2 \models t_1[x \to y] = t_2[x \to y]. \qquad \square$$

Further on we assume that for the identities of the form (1) we have $n \geq m \geq 0,\ n > 1$.

Lemma 1. *For any identity of the form* (1) *we have*

$$|var(t_1)| \leq n,\ |var(t_2)| \leq m,\ |var(t_1 = t_2)| \leq m + n,$$
$$var(t_2) \subseteq var(t_1) \Rightarrow |var(t_1 = t_2)| \leq n.$$

$\square$

Lemma 2. *If $|var(t_1 = t_2)| > n$ then there are different variables $x, y \in X$ such that $x \in var(t_1) \setminus var(t_2)$, $y \in var(t_2) \setminus var(t_1)$.*

Proof. $var(t_1) \subseteq var(t_2) \Rightarrow var(t_1 = t_2) = var(t_2)$ and $var(t_2) \subseteq var(t_1) \Rightarrow var(t_1 = t_2) = var(t_1)$. $\square$

Lemma 3. *If $|var(t_1 = t_2)| > n$, then the identity $t_1 = t_2$ is equivalent to the system Δ of two identities, where*

$$\Delta = \{t_1 = t_2[y \to x],\ t_2[y \to x] = t_2\}$$

for any $x \in var(t_1) \setminus var(t_2)$, $y \in var(t_2) \setminus var(t_1)$.

Proof. By *Proposition 1* we have $t_1 = t_2 \models t_1 = t_2[y \to x]$ and by transitivity it follows that $t_1 = t_2,\ t_1 = t_2[y \to x] \models t_2 = t_2[y \to x]$. Thus $t_1 = t_2 \models \Delta$. The converse is obvious. $\square$

Lemma 4. *For the identities of Δ we have :*

$$|var(t_1 = t_2[y \to x])| = |var(t_1 = t_2)| - 1,$$

$$|var(t_2[y \to x] = t_2)| = |var(t_2)| + 1. \qquad \square$$

Lemma 5. *Let $n = m$ and $|var(t_1 = t_2)| = n + 1$. Then there are variables x, y, z such that $x \in var(t_1) \setminus var(t_2)$, $y \in var(t_2) \setminus var(t_1)$, $z \in var(t_1 = t_2)$, $z \neq x, y$, and $t_1 = t_2$ is equivalent to the system Λ of three identities, where*

$$\Lambda = \{t_1 = t_2[y \to z],\ t_1[x \to y] = t_2[y \to z],\ t_1[x \to y] = t_2\}.$$

Proof. By *Lemma 2* we have the existence of variables x, y and the existence of z follows from the assumption $n > 1$. It is clear that $\Lambda \models t_1 = t_2$. By *Proposition 1*, we have $t_1 = t_2 \models t_1 = t_2[y \to z]$ and $t_1 = t_2 \models t_1[x \to y] = t_2$. Let $t_3 \equiv t_2[y \to z]$. Then $x, y \notin var(t_3)$ and $t_1 = t_2 \models t_1 = t_3$. By *Proposition 1* we have $t_1 = t_3 \models t_1[x \to y] = t_3$, i.e. $t_1 = t_3 \models t_1[x \to y] = t_2[y \to z]$, which implies $t_1 = t_2 \models t_1[x \to y] = t_2[y \to z]$ as well. Thus, $t_1 = t_2 \models \Lambda$. □

Lemma 6. *For the identities of* Λ *we have :*

$$|var(t_1 = t_2[y \to z])| = |var(t_1[x \to y] = t_2[y \to z])| = \\ = |var(t_1[x \to y] = t_2)| = n.$$

□

Lemma 5' *Let* $n = m$ *and* $|var(t_1 = t_2)| = n + 1$, *and let there be variables* x, y, z *such that* $x, y \in var(t_1) \setminus var(t_2)$, $z \in var(t_2) \setminus var(t_1)$, $x \neq y$. *Then the identity* $t_1 = t_2$ *is equivalent to the system* Λ' *of two identities, where :*

$$\Lambda' = \{t_1 = t_2[z \to x],\ t_1 = t_2[z \to y]\},$$

and

$$|var(t_1 = t_2[z \to x])| = |var(t_1 = t_2[z \to y])| = n.$$

Proof. It is clear that $t_1 = t_2 \models \Lambda'$. Conversely, $\Lambda' \models t_3 \equiv t_1[y \to u] = t_2[z \to x]$ for a new variable $u \notin var(t_1 = t_2)$. Then, since $\Lambda' \models t_2[z \to x] = t_2[z \to y]$, we have $\Lambda' \models t_3 = t_2[z \to y] \equiv t_4$, which implies $\Lambda' \models t_3 = t_4[y \to z] \equiv t_2$. Finally, $\Lambda' \models t_3[u \to y] = t_2$ and $t_3[u \to y] \equiv t_1$, i.e. $\Lambda' \models t_1 = t_2$. □

A system Σ of finitely many identities of the form (1) is called *reduced* if for each identity $t_1(x_{i_1}, ..., x_{i_n}) = t_2(x_{j_1}, ..., x_{j_m}) \in \Sigma$, such that $n \geq m$, we have $|var(t_1 = t_2)| \leq n$.

Theorem 1. *Every identity of the form* (1), *where* $n > 1$, $n \geq m \geq 0$, *is equivalent to a finite reduced system* Σ *of identities, and* Σ *can be obtained effectively from* (1).

Proof. If an identity $t_1 = t_2$ of the form (1) is not already reduced then, by *Lemma 3*, it is equivalent to the system Δ. By *Lemma 4*, $t_1 = t_2[y \to x] \in \Delta$ has one variable less than $t_1 = t_2$, and if it happens that $|var(t_2[y \to x] = t_2)| = |var(t_2)| + 1 = n + 1$ for $t_2[y \to x] = t_2 \in \Delta$, then by *Lemma 5* (or *Lemma 5'*) and *Lemma 6*, we have that the identity $t_2[y \to x] = t_2$ is equivalent to a corresponding reduced system of identities of the form given by Λ (where instead of $t_1 = t_2$ we consider the identity $t_2[y \to x] = t_2$). Further, the procedure is repeated for the identity $t_1 = t_2[y \to x]$. □

Example 1 We will show that the reduction of the number of variables in identities can be useful for obtaining structural informations of some algebraic objects. Consider the identity

$$xyzu = zvw \tag{3}$$

on semigroups, where x, y, z, u, v, w are different variables. We can apply *Lemma 3* to get the equivalent system:

$$xyzu = zvx, \tag{4}$$
$$zvx = zvw. \tag{5}$$

By *Lemma 5*, (5) is eqivalent to the reduced identity:

$$zvx = zvz. \tag{6}$$

Again by *Lemma 3*, (4) is equivalent to the system:

$$xyzu = zux, \tag{7}$$
$$zux = zvx. \tag{8}$$

Clearly, (7) is reduced, and by *Lemma 5*, (8) reduces to the identity

$$zvx = zxx. \tag{9}$$

Consequently, (3) is equivalent to the reduced system consisting of (6), (7) and (9).

The obtained reduced system can be used as follows. By (6) we can conclude that the product xyz depends on x and y only. Similarly, (9) tells us that the same product depends only on x and z. Therefore, xyz depends only on x, i.e. $xyz = \theta(x)$ for some function θ. Then we have $\theta(z) = zux$, $\theta(x)u = xyzu$, and (7) implies that $\theta(z) = \theta(x)u = const$.

We conclude that the semigroups satisfying the identity (3) have the property

$$xyz = 0, \quad x0 = 0x = 0$$

where 0 is a fixed element, in fact a zero. The converse is obvious. □

Example 2 Let $f, h \in \Omega$ for the ternary operation symbol f and 4-ary h. Let x, y, z, t, u, v be different variables and consider the identity

(10) $$h(x,y,z,x) = f(t,u,v).$$

Then (10) is equivalent to the following reduced system :

(11) $$h(x,y,z,x) = f(x,z,t),$$
(12) $$f(x,y,z) = f(z,y,z),$$
(13) $$f(x,y,z) = f(x,x,z).$$

Moreover, by *Lemma 3*, we can replace the identity (11) by these two identities:

(11') $$h(x,y,z,x) = f(x,z,z),$$
(11") $$f(x,z,z) = f(x,z,y).$$ □

Remarks It is clear how one can apply *Theorem 1* for a system of finitely many identities of the form (1). Moreover, the algorithm given in *Theorem 1* can be applied for any recursively defined system of identities with a bounded number of variables appearing in it. Also, in some cases, *Lemma 3* allows us to reduce the number of variables significantly below n. Such a situation we have in the above example, where the identity (11) is eqivalent to the system of two identities $\{(11'),(11'')\}$ containing one variable less than in (11). Then we may say that the reduced system $\{(11'),(11''),(12),(13)\}$ is "more reduced" then the reduced system $\{(11),(12),(13)\}$. An open question is, how far can we go in the reductions? This gives rise to the following problem:

Let us have an identity of the form (1) *such that* $var(t_1) = var(t_2) \cup \{x\}$, *where* $x \notin var(t_2)$. *Is it true that* (1) *is not equivalent to any system of identities containing* $\leq |var(t_2)|$ *variables?*

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