

FREE GROUPOIDS WITH $x^n = x$

Đorđi Čupona*, Naum Celakoski**

*MANU, P. O. Box 428, 91000 Skopje

**Mašinski fakultet, 91000 Skopje

A b s t r a c t: We give a description of the class of free objects in the variety \mathcal{U}_n of groupoids which satisfy the law $x^n = x$. In Section 0 we state necessary preliminary definitions and the main results of the paper, which are shown in Sections 1-3. Corresponding results concerning free objects in commutative \mathcal{U}_n -groupoids are stated in Section 4.

0. INTRODUCTION

A *groupoid* is an algebra $G = (G, \cdot)$, where: $(x, y) \mapsto xy$ is a binary operation on G . Notions as subgroupoids, homomorphisms and variety of groupoids have usual meanings. If \mathcal{U} is a variety of groupoids, then “ G is a \mathcal{U} -groupoid”, “ G is a \mathcal{U} -free groupoid”, “ G is a free groupoid”, means “ $G \in \mathcal{U}$ ”, “ $G \in \mathcal{U}$ and G is free in \mathcal{U} ”, “ G is free in the variety of all groupoids”, respectively.

In this paper we assume that n is a given integer such that $n \geq 2$, and \mathcal{U}_n is the variety of groupoids which satisfies the law $x^n = x$, where:

$$x^1 = x, x^k x = x^{k+1}.$$

Let $G = (G, \cdot) \in \mathcal{U}_n$. An element $a \in G$ is called \mathcal{U}_n -prime in G iff the following implication holds in G

$$a = bc \Rightarrow b = c^{n-1} \tag{0.1}$$

Thus, $a \in G$ is *not* \mathcal{U}_n -prime in G iff there exist $b, c \in G$ which satisfy the following statement:

$$a = bc \ \& \ b \neq c^{n-1}. \tag{0.1'}$$

Then we say that b and c are \mathcal{U}_n -divisors of a in G . A (finite or infinite) sequence a_1, a_2, \dots of elements in G is called a \mathcal{U}_n -divisor chain in G iff $a_{k+1} | a_k^{(1)}$ for all $k = 1, 2, \dots$; we assume that, for any $a \in G$, the (one-element) sequence a is a \mathcal{U}_n -divisor chain in G . And, G is \mathcal{U}_n -injective iff, for any $a, b, c, d \in G$, the following implication holds:

$$ab = cd \ \& \ a \neq b^{n-1} \ \& \ c \neq d^{n-1} \Rightarrow a = c \ \& \ b = d. \quad (0.2)$$

Now we state the main results.

Theorem 1. Let B be a non-empty set and $F = (F, \cdot)$ a free groupoid with the basis B . Let R , a subset of F , and $R = (R, *)$ be defined as follows:

$$B \subseteq R \text{ and } u, v \in F \Rightarrow \{uv \in R \Leftrightarrow u, v \in R \ \& \ u \neq v^{n-1}\}, \quad (0.3)$$

$$u, v \in R \Rightarrow u * v = \begin{cases} uv, & \text{if } uv \in R \\ v, & \text{if } u = v^{n-1} \end{cases} \quad (0.4)$$

Then R is a \mathcal{U}_n -free groupoid and B is the unique \mathcal{U}_n -basis of R ²⁾

Theorem 2. If $H = (H, \cdot)$ is \mathcal{U}_n -injective, then the following conditions are equivalent:

- (i) H is \mathcal{U}_n -free.
- (ii) There is a mapping $| \cdot | : a \mapsto |a|$ from H into the set of positive integers, such that: $b | a \Rightarrow |b| < |a|$.
- (iii) Every \mathcal{U}_n -divisor chain in H is finite.
- (iv) The set B of \mathcal{U}_n -primes in H is a generating set of H .

Theorem 3. The class of \mathcal{U}_n -free groupoids is hereditary. If k is a positive integer such that $\max\{n-1, k\} \geq 2$, then in any \mathcal{U}_n -free groupoid with the rang k there exist subgroupoids with infinite rang.

This will be shown in Section i (i = 1, 2, 3), and corresponding results for the variety of commutative \mathcal{U}_n -groupoids will be stated in Section 4.

Remark. It is well known that there exist $\alpha_n = (2n-2)! / n!(n-1)!$ possibilities of defining n -th powers in groupoids (see, for example: [2], III.2 or [4], I.4). Therefore, there exist α_n varieties of groupoids each of which is defined by an axiom of the form $x^n = x$. This suggests the question whether the corresponding T.1- T.3 are satisfied in the general case.

¹⁾ $d | a$ means " d is a \mathcal{U}_n -divisor of a ".

²⁾ If $n=2$ and $|B|=1$, then $R=B$. If $n \geq 3$ or $|B| \geq 2$, then B is a proper subset of R , and R is infinite.

1. A CANONICAL DESCRIPTION OF FREE \mathcal{U}_n -GROUPOIDS

It is clear that the class \mathcal{U}_1 (of groupoids with $x^1 = x$) is the variety of groupoids. Therefore the notions of " \mathcal{U}_1 -prime" and " \mathcal{U}_1 -injectivity" have the meanings of "prime" and "injectivity" in the class of groupoids. Namely, $a \in G$ is *prime* in a groupoid $G = (G, \cdot)$ iff $a \neq bc$ for any $b, c \in G$, and G is *injective* iff $ab = cd$ implies $a = c, b = d$, for any $a, b, c, d \in G$. Thus we obtain the following *characterization of free groupoids*, i.e. free \mathcal{U}_1 -groupoids ([1], I.1).

Proposition 1.1. *A groupoid $F = (F, \cdot)$ is free with the basis B iff F is injective, B is the set of primes in F and generates F .*

We will use the following properties of free groupoids as well.

Proposition 1.2. *If F is a free groupoid with the basis B , then the following statements hold.*

1) *There exists a unique mapping $|\cdot| : u \mapsto |u|$, from F into the set of positive integers such that:*

$$|b| = 1, |uv| = |u| + |v| \quad (1.1)$$

for any $b \in B, u, v \in F$. (We say that $|u|$ is the *norm* of u .)

2) *If $k, m \geq 1, u, v \in F$, then: $u^m = v^m \Leftrightarrow u = v$, and $u^k = u^m \Leftrightarrow k = m$.*

By induction on norm, the following statement can be easily shown.

Proposition 1.3. *The set R and the groupoid \mathbf{R} are well defined by (0.3) and (0.4) - respectively; and, B is the least generating subset of \mathbf{R} . If $n = 2$ and $|B| = 1$, then $R = B$; and, if $n \geq 3$ or $|B| \geq 2$, then R is infinite and a proper subset of F .*

Proposition 1.4. *$\mathbf{R} \in \mathcal{U}_n$, and B is the set of \mathcal{U}_n -primes in \mathbf{R} .*

Proof. Denote by u_*^k the corresponding k -th power of $u \in R$, i.e. $u \in R \Rightarrow u_*^1 = u, u_*^{k+1} = u_*^k * u$. Assume that $1 \leq m < n-1, u_*^m = u^m$, for any $u \in R$. Then: $u_*^{m+1} = u_*^m * u = u^m u = u^{m+1}$. This implies that $u_*^{n-1} = u^{n-1}$, and thus: $u_*^n = u_*^{n-1} * u = u^{n-1} * u = u$. Thus $\mathbf{R} \in \mathcal{U}_n$. Clearly, an element $u \in R$ is \mathcal{U}_n -prime in \mathbf{R} iff $u \in B$. \square

Proposition 1.5. *\mathbf{R} is a \mathcal{U}_n -free groupoid with the basis B .*

Proof. Let $G = (G, \cdot) \in \mathcal{U}_n$, and $\lambda: B \rightarrow G$ be an arbitrary mapping. Denote by φ the homomorphism from F into G which extends λ , and let $\psi = \varphi|_R$ be the restriction of φ on R . Thus ψ extends λ , and it can be easily shown that $\psi: R \rightarrow G$ is a homomorphism. Thus, \mathbf{R} is \mathcal{U}_n -free and B is the basis of \mathbf{R} . \square

This completes the proof of Th.1.

The following statements can be also easily shown, and we will use them in the proof of Th.2.

Proposition 1.6. R is \mathcal{U}_n -injective.

Proposition 1.7. If u is a \mathcal{U}_n -divisor of v in R , then $|u| < |v|$.³⁾

Proposition 1.8. Every \mathcal{U}_n -divisor chain in R is finite.

2. PROOF OF THEOREM 2

From Pr.1.3-1.8 we obtain the following “first half” of Th.2.

Proposition 2.1. If $H = (H, \cdot)$ is a \mathcal{U}_n -free groupoid with the basis B , then:

- 1) H is \mathcal{U}_n -injective.
- 2) There is a mapping $||: u \mapsto |u|$ from H into the set of positive integers such that: $u | v \Rightarrow |u| < |v|$.
- 3) Each \mathcal{U}_n -divisor chain in H is finite.
- 4) B is the set of \mathcal{U}_n -primes in H .

Therefore:

Proposition 2.2. If H satisfies the condition (i) of Th.2, then it satisfies (ii), (iii), (iv) as well.

The following more general statements will imply “the second part” of Th.2.

Proposition 2.3. Let $H = (H, \cdot)$ be a \mathcal{U}_n -injective groupoid. Then $a \in H$ is not \mathcal{U}_n -prime in H iff there exists a unique pair $(b, c) \in H^2$ such that: $a = bc$, $b \neq c^{n-1}$. Moreover, in this case: $n \geq 3$ and $a = b^2$ implies $b \neq a$.

Proposition 2.4. If $H \in \mathcal{U}_n$ and H satisfies the condition (ii) of Th.2, then it satisfies (iii) as well.

Proposition 2.5. Assume that: $H \in \mathcal{U}_n$, every \mathcal{U}_n -divisor chain in H is finite, and the set of \mathcal{U}_n -divisors of a is finite, for every $a \in H$. Then the following statements hold:

- a) The set B of \mathcal{U}_n -primes in H is non-empty.
- b) There is a mapping $||: u \mapsto |u|$ from H into the set of positive

³⁾ $||: R \rightarrow \mathbb{N}$ is the restriction of $||: F \rightarrow \mathbb{N}$ (see Pr.1.2).

integers such that $|b| = 1$, for each $b \in B$, and $c | d \Rightarrow |c| < |d|$ for any $a, b \in H$.

Proof. By König Lemma (see, for example, [3],4.), the set of \mathcal{U}_n -divisor chains in H , with a given first member is finite. Let $a \in H$, and let $|a|$ be the largest possible length of \mathcal{U}_n -divisor chains in H with the first member a . Then we have: $|a| = 1$ iff a is \mathcal{U}_n -prime in H and $c | a \Rightarrow |c| < |a|$. \square

Proposition 2.6. *If H is a \mathcal{U}_n -injective groupoid which satisfies (iii) of Th.2, then it satisfies (iv) as well.*

Proof. By Pr.2.5, the set B of \mathcal{U}_n -primes in H is non-empty, and there exists a mapping $|| : H \rightarrow \mathbb{N}$, such that: $b \in B \Leftrightarrow |b| = 1$, and $c | a \Rightarrow |c| < |a|$. Denote by Q the subgroupoid of H generated by B . Thus $b \in H$ & $|b| = 1 \Rightarrow b \in Q$. Assume that: $c \in H$ & $|c| \leq k \Rightarrow c \in Q$, and let $a \in H$, $|a| = k+1$. Then by Pr.2.3, there exists a unique pair $b, c \in H$, such that $a = bc$, $b \neq c^{n-1}$ and $|b|, |c| \leq k$. Thus $b, c \in Q$, which implies $a = bc \in Q$. \square

The following statement will complete the proof of Th.2.

Proposition 2.7. *Let H be a \mathcal{U}_n -injective groupoid and the set B of \mathcal{U}_n -primes in H generates H . Then H is a \mathcal{U}_n -free groupoid with the basis B .*

Proof. First we have $H = \bigcup \{ B_n \mid n \geq 1 \}$, where $B_1 = B$ and $B_{k+1} = B_k \cup \{ uv \mid u, v \in B_k \}$. By Pr.2.3, $u \in B_{k+1} \setminus B_k$ iff there exists a unique pair $(v, w) \in B_k^2$ such that: $u = vw$, $v \neq w^{n-1}$, $v, w \in B_k$ and $(v \notin B_{k-1} \text{ or } w \notin B_{k-1})$.

Let $G = (G, \cdot) \in \mathcal{U}_n$, and $\lambda: B \rightarrow G$ be a mapping. There exists a unique sequence of mappings: $(\varphi_k : B_k \rightarrow G \mid k \geq 1)$ defined in the following way: 1) $\varphi_1 = \lambda$; 2) φ_{m+1} is an extension of φ_m ; 3) If $u \in B_{k+1} \setminus B_k$ and $u = vw$, where $v \neq w^{n-1}$, then $\varphi_{k+1}(u) = \varphi_k(v) \varphi_k(w)$. Therefore, there exists a unique mapping $\varphi: H \rightarrow G$ such that: $u \in B_k \Rightarrow \varphi(u) = \varphi_k(u)$, and moreover $\varphi: H \rightarrow G$ is a homomorphism which extends λ . \square

3. SUBGROUPOIDS OF \mathcal{U}_n -FREE GROUPOIDS

The following statements are clear.

Proposition 3.1. *If a groupoid H is \mathcal{U}_n -injective, then any subgroupoid of H is \mathcal{U}_n -injective as well*

Proposition 3.2. *If Q is a subgroupoid of a \mathcal{U}_n -groupoid H , and if b is a \mathcal{U}_n -divisor of a in Q , then b is a \mathcal{U}_n -divisor of a in H , as well.*

Proposition 3.3. *If a \mathcal{U}_n -groupoid H satisfies the condition (iii) of Th.2, and if Q is a subgroupoid of H , then Q satisfies (iii), as well.*

As a corollary, we obtain the following “first part” of Th.3.

Proposition 3.4. *If H is a \mathcal{U}_n -free groupoid, and Q is a subgroupoid of H , then Q is \mathcal{U}_n -free, as well.*

As concerns the “second part”, we will consider two cases: $n \geq 3$ and $n = 2$.

Proposition 3.5. *Let $n \geq 3$ and H be a free \mathcal{U}_n -groupoid with the one-element basis $B = \{b\}$, and let $C = \{c_k \mid k \geq 1\} \subset H$ be defined as follows:*

$$c_1 = b^2, c_{k+1} = c_k b. \quad (3.1)$$

Then the rank of the subgroupoid Q of H , generated by C is infinite.

Proof. It can be easily seen that $c_i = c_j \Rightarrow i = j$, and that C is the set of \mathcal{U}_n -primes in Q . \square

Proposition 3.6. *Let $H = (H, \cdot)$ be a \mathcal{U}_2 -free groupoid with two-elements basis $B = \{a, b\}$, and let $C = \{c_k \mid k \geq 1\} \subset H$ be defined as follows:*

$$c_1 = ab, c_{k+1} = c_k b. \quad (3.2)$$

Then C is the basis of the subgroupoid Q , generated by C , and C is infinite.

4. COMMUTATIVE \mathcal{U}_n -GROUPOIDS

Denote by S_n the variety of commutative \mathcal{U}_n -groupoids and assume that $n \geq 2$. Let $G = (G, \cdot) \in S_n$. We say that G is S_n -injective iff

$$(\forall a, b)(a \neq b^{n-1}, b \neq a^{n-1}, c \neq d^{n-1}, d \neq c^{n-1}, ab = cd \Rightarrow \{a, b\} = \{c, d\}). \quad (4.1)$$

An element $a \in G$ is S_n -prime in G iff

$$a = bc \Rightarrow b = c^{n-1} \text{ or } c = b^{n-1}. \quad (4.2)$$

Thus $a \in G$ is not S_n -prime in G iff there exist $b, c \in G$ such that

$$a = bc \ \& \ b \neq c^{n-1} \ \& \ c \neq b^{n-1}. \quad (4.3)$$

Then b and c are called S_n -divisors of a in G .

The results obtained above for \mathcal{U}_n -groupoids can be “translated” for S_n -groupoids. Here we will state only the following *canonical description of free S_n -groupoids*.

Theorem 1'. Let $F = (F, \cdot)$ be a free commutative groupoid with the basis B . Let R , a subset of F , and $R = (R, *)$ be defined as follows: $B \subseteq R$ and

$$u, v \in F \Rightarrow (uv \in R \Leftrightarrow u, v \in R \ \& \ u \neq v^{n-1} \ \& \ v \neq u^{n-1}), \quad (4.4)$$

$$u, v \in R \Rightarrow u * v = \begin{cases} uv, & \text{if } uv \in R \\ v, & \text{if } u = v^{n-1} \\ u, & \text{if } v = u^{n-1} \end{cases}$$

Then $R = (R, *)$ is S_n -free groupoid with the basis B .

(Th.2' and Th.3' can be obtained from Th.2 and Th.3 respectively, by replacing \mathcal{U}_n with S_n . We will not state them explicitly.)

REFERENCES

- [1] R.H.Bruck: *A Survey of Binary Systems*, Berlin-Göttingen-Heidelberg, 1956
- [2] P.M.Cohn: *Universal algebra*, Harper & Row, London 1965
- [3] J.A.Robinson: *Logic Form and Function*, North Holland Co, 1979
- [4] S.Markovski: *Konečna matematika*, Skopje 1993

СЛОБОДНИ ГРУПОИДИ СО $x^n = x$

Г. Чупона, Н. Целакоски
Резиме

Во работава се дава опис на класата слободни објекти во многуобразието \mathcal{U}_n од групоиди коишто го задоволуваат идентитетот $x^n = x$, каде што n е даден природен број. Главните резултати се формулираат во §0, а се образложуваат во §1-§3. Соодветни резултати во врска со комутативни \mathcal{U}_n -групоиди се наведени во §4.